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# A Paley–Wiener theorem for the $\Theta$ -hypergeometric transform: the even multiplicity case <sup>☆</sup>

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## Abstract

The  $\Theta$ -hypergeometric functions generalize the spherical functions on Riemannian symmetric spaces and the spherical functions on non-compactly causal symmetric spaces. In this paper we consider the case of even multiplicity functions. We construct a differential shift operator  $D_m$  with smooth coefficients which generates the  $\Theta$ -hypergeometric functions from finite sums of exponential functions. We then use this fact to prove a Paley–Wiener theorem for the  $\Theta$ -hypergeometric transform.

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## Résumé

Les fonctions  $\Theta$ -hypergéométriques généralisent les fonctions sphériques sur les espaces symétriques riemanniens et les fonctions sphériques sur les espaces symétriques non-compactement causaux. Dans cet article nous considérons le cas de fonctions de multiplicité paire. Nous construisons un opérateur différentiel de décalage  $D_m$  à coefficients indéfiniment différentiables qui donne les fonctions  $\Theta$ -hypergéométriques à partir de sommes finies de fonctions exponentielles. Nous utilisons ensuite cette construction pour établir un théorème de Paley–Wiener pour la transformation  $\Theta$ -hypergéométrique.

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## 1. Introduction

Let  $\mathfrak{a}$  be an  $l$ -dimensional Euclidean real vector space with complexification  $\mathfrak{a}_{\mathbb{C}}$ , and let  $\mathfrak{a}^*$  and  $\mathfrak{a}_{\mathbb{C}}^*$  respectively denote the real and complex dual vector spaces of  $\mathfrak{a}$ . For a compact subset  $E$  of  $\mathfrak{a}$  let  $\text{conv}(E)$  denote its closed convex hull (i.e., the intersection of all closed half-spaces in  $\mathfrak{a}$  containing  $E$ ). The support function of  $E$  is the function  $q_E : \mathfrak{a}^* \rightarrow \mathbb{R}$  defined by:

$$q_E(\lambda) := \sup_{H \in E} \lambda(H) = \sup_{H \in \text{conv}(E)} \lambda(H). \quad (1)$$

Let  $C$  be a compact convex subset of  $\mathfrak{a}$ , and let  $C_c^\infty(C)$  denote the space of smooth functions on  $\mathfrak{a}$  with support contained in  $C$ . The Paley–Wiener space  $\text{PW}(C)$  consists of the entire functions  $g : \mathfrak{a}_{\mathbb{C}}^* \rightarrow \mathbb{C}$  which are of exponential type  $C$  and rapidly decreasing, i.e., for every  $N \in \mathbb{N}$  there is a constant  $C_N \geq 0$  such that

$$|g(\lambda)| \leq C_N (1 + |\lambda|)^{-N} e^{q_C(\text{Re } \lambda)}$$

for all  $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ . The Euclidean Fourier transform of a sufficiently regular function  $f : \mathfrak{a} \rightarrow \mathbb{C}$  is the function  $\mathcal{F}f : \mathfrak{a}_{\mathbb{C}}^* \rightarrow \mathbb{C}$  defined by:

$$\mathcal{F}f(\lambda) := \int_{\mathfrak{a}} f(H) e^{\lambda(H)} dH,$$

where  $dH$  denotes the Lebesgue measure on  $\mathfrak{a}$ . A classical theorem due to Paley and Wiener characterizes  $\text{PW}(C)$  as the image of  $C_c^\infty(C)$  under the Euclidean Fourier transform (see, e.g., [16], Theorem 7.3.1, or [18], Theorem 8.3 and Proposition 8.6).

**Theorem 1.1** (Paley–Wiener). *Let  $C$  be a compact convex subset of  $\mathfrak{a}$ . Then the Euclidean Fourier transform maps  $C_c^\infty(C)$  bijectively onto  $\text{PW}(C)$ . Moreover, if  $C$  is stable under the action of a finite group  $W$  of linear automorphisms of  $\mathfrak{a}$ , then the Fourier transform maps the subspace  $C_c^\infty(C)^W$  of  $W$ -invariant elements in  $C_c^\infty(C)$  onto the subspace  $\text{PW}(C)^W$  of  $W$ -invariant elements in  $\text{PW}(C)$ .*

Suppose that  $A$  is a connected simply connected Abelian Lie group with Lie algebra  $\mathfrak{a}$ . Then  $\exp : \mathfrak{a} \rightarrow A$  is a diffeomorphism. Denote by  $\log$  the inverse of  $\exp$ . The Euclidean Fourier transform of a sufficiently regular functions  $f : A \rightarrow \mathbb{C}$  is the function  $\mathcal{F}_A f : \mathfrak{a}_{\mathbb{C}}^* \rightarrow \mathbb{C}$  defined by:

$$\mathcal{F}_A f(\lambda) := \int_A f(a) e^{\lambda(\log a)} da,$$

where the Haar measure  $da$  on  $A$  is the pullback under the exponential map of the Haar measure  $dH$  on  $\mathfrak{a}$ . Let  $W$  be a finite group acting on  $\mathfrak{a}$  by linear automorphism. Then we define an action by  $W$  on  $A$  by  $w(\exp H) = \exp w(H)$ . Denote by  $C_c^\infty(C)$  the space of

smooth functions on  $A$  with support in the compact set  $\exp C$ , and let  $C_c^\infty(C)^W$  be the subspace of  $W$ -invariant elements. Composition with  $\exp$  and Theorem 1.1 prove that the Euclidean Fourier transform  $\mathcal{F}_A$  is a bijection of  $C_c^\infty(C)$  onto  $\text{PW}(C)$  and restricts to a bijection of  $C_c^\infty(C)^W$  onto  $\text{PW}(C)^W$ .

In this paper we generalize Theorem 1.1 to the case of the  $\Theta$ -hypergeometric transform of [31] and [33] corresponding to a triple  $(\mathfrak{a}, \Sigma, m)$ . Here  $\mathfrak{a}$  is a Euclidean space,  $\Sigma$  an irreducible reduced root system in the dual space  $\mathfrak{a}^*$  of  $\mathfrak{a}$ , and  $m$  a positive even multiplicity function on  $\Sigma$  (see Section 2 for the precise definitions). Furthermore,  $\Theta$  denotes a subset of the fundamental system  $\Pi$  of simple roots in a fixed set  $\Sigma^+$  of positive roots in  $\Sigma$ .

Let  $W_\Theta$  denote the parabolic subgroup of  $W$  generated by the reflections  $r_\alpha$  with  $\alpha \in \Theta$ . With each choice of  $\Theta$  and  $\lambda \in \mathfrak{a}_\mathbb{C}^*$  is associated a function  $\varphi_\Theta(m; \lambda, a)$ , which is defined for  $a = \exp H$  with  $H$  in a certain  $W_\Theta$ -invariant open cone  $\mathfrak{a}_\Theta$  in  $\mathfrak{a}$ . We set  $A_\Theta := \exp(\mathfrak{a}_\Theta)$ . The function  $\varphi_\Theta(m; \lambda)$  is called the  $\Theta$ -hypergeometric function of spectral parameter  $\lambda$ . It is a joint eigenfunction of the hypergeometric system of differential operators of spectral parameter  $\lambda$  constructed by Heckman and Opdam. It is real analytic and  $W_\Theta$ -invariant. As a function of  $\lambda \in \mathfrak{a}_\mathbb{C}^*$ ,  $\varphi_\Theta(m; \lambda, a)$  is meromorphic with simple poles located along a specific finite family of affine complex hyperplanes. Indeed, there is a polynomial function  $e_\Theta^-(m; \lambda)$ , which is a finite product of affine functions of  $\lambda$ , and there is a tubular neighborhood  $U_\Theta$  of  $A_\Theta$  in the complexification  $A_\mathbb{C}$  of  $A$  so that  $e_\Theta^-(m; \lambda)\varphi_\Theta(m; \lambda, a)$  extends as a holomorphic function of  $(\lambda, a) \in \mathfrak{a}_\mathbb{C}^* \times U_\Theta$ . We refer the reader to Theorem 3.6 below for detailed information on the regularity properties of the  $\Theta$ -hypergeometric functions.

The  $\Theta$ -hypergeometric transform of a  $W_\Theta$ -invariant function  $f$  on  $A_\Theta$  is the  $W_\Theta$ -invariant function  $\mathcal{F}_\Theta f(m)$  on  $\mathfrak{a}_\mathbb{C}^*$  defined for  $\lambda \in \mathfrak{a}_\mathbb{C}^*$  by:

$$\mathcal{F}_\Theta f(m; \lambda) := \frac{1}{|W_\Theta|} \int_{A_\Theta} f(a) \varphi_\Theta(m; \lambda, a) \Delta(m; a) da, \quad (2)$$

provided the integral converges. Here

$$\Delta(m) := \prod_{\alpha \in \Sigma^+} (e^\alpha - e^{-\alpha})^{m_\alpha}. \quad (3)$$

The  $\Theta$ -hypergeometric transform contains important special cases:

- (a) For  $m = 0$ , the  $\Theta$ -hypergeometric function of spectral parameter  $\lambda$  is  $\varphi_\Theta(m; \lambda, a) = \sum_{w \in W_\Theta} e^{w\lambda(\log a)}$ . The  $\Theta$ -hypergeometric transform of a  $W_\Theta$ -invariant function on  $A_\Theta$  therefore coincides with its Euclidean Fourier transform.
- (b) If  $\Theta = \Pi$ , then the  $\Theta$ -hypergeometric transform agrees with the Opdam transform [28], and hence with the spherical transform of Harish-Chandra when the triple  $(\mathfrak{a}, \Sigma, m)$  corresponds to a Riemannian symmetric space  $G/K$  of noncompact type. The condition on the even multiplicities singles out the spaces  $G/K$  with the property that all Cartan subalgebras in the Lie algebra  $\mathfrak{g}$  of  $G$  are conjugate under the adjoint group of  $\mathfrak{g}$  (cf. [12], p. 429; see also Section 2 below).

- (c) When  $(\mathfrak{a}, \Sigma, m)$  corresponds to a Hermitian symmetric space  $G/K$  and  $\Theta$  is the set  $\Pi_0$  of simple positive compact roots, then the  $\Theta$ -hypergeometric transform coincides with the spherical Laplace transform on the non-compactly causal space  $G/H$  having  $G/K$  as Riemannian dual. See Section 2.

**Definition 1.2** (*Paley–Wiener space*). Let  $m \in \mathcal{M}^+$  be an even multiplicity function, and let  $\Theta \subset \Pi$  be a fixed set of positive simple roots. Let  $C$  be a compact, convex and  $W_\Theta$ -invariant subset of  $\mathfrak{a}_\Theta$ . The *Paley–Wiener space*  $\text{PW}_\Theta(m; C)$  is the space of all  $W_\Theta$ -invariant meromorphic functions  $g: \mathfrak{a}_\mathbb{C}^* \rightarrow \mathbb{C}$  satisfying the following properties:

- (1)  $e_\Theta^-(m; \lambda)g(\lambda)$  is a rapidly decreasing entire function of exponential type  $C$ , that is for every  $N \in \mathbb{N}$  there is a constant  $C_N \geq 0$  such that

$$|e_\Theta^-(m; \lambda)g(\lambda)| \leq C_N (1 + |\lambda|)^{-N} e^{q_C(\text{Re } \lambda)}$$

for all  $\lambda \in \mathfrak{a}_\mathbb{C}^*$ .

- (2) The function

$$\text{P}_\Theta^{\text{av}} g(\lambda) := \sum_{w \in W_\Theta \setminus W} g(w\lambda) \quad (4)$$

extends to an entire function on  $\mathfrak{a}_\mathbb{C}^*$ .

By  $W_\Theta$ -invariance of  $g$ , Condition (2) in Definition 1.2 is equivalent to

- (2') The function

$$\text{P}^{\text{av}} g(\lambda) := \frac{1}{|W|} \sum_{w \in W} g(w\lambda) \quad (5)$$

extends to an entire function on  $\mathfrak{a}_\mathbb{C}^*$ .

Condition (2) is automatically satisfied in the Euclidean case  $m = 0$ , in the complex case  $m = 2$ , and when  $\Theta = \Pi$  (see Proposition 6.1). Notice that  $e_\Theta^- \equiv 1$  in the Euclidean case and when  $\Theta = \Pi$ . Therefore  $\text{PW}_\Theta(C; 0) \equiv \text{PW}(C)^{W_\Theta}$  and  $\text{PW}_\Pi(C; m) \equiv \text{PW}(C)^W$ .

A generalization of the classical Paley–Wiener theorem to the  $\Theta$ -hypergeometric transform holds under certain restrictions on  $\Theta$  and  $m$ , which we indicate as Condition A. It requires either that there are “not too many”  $\lambda$ -singular hyperplanes (condition on  $m$ ) or that the cone  $\mathfrak{a}_\Theta$  is “wide enough” in  $\mathfrak{a}$  (condition on  $\Theta$ ). We refer to Section 12 for the precise statement of Condition A. Here we only observe that all pairs  $(\Theta, m)$  with  $\Theta = \Pi$  or  $m = 0, 2$  or corresponding to a  $K_\varepsilon$  symmetric space with even multiplicities satisfy this condition. The  $K_\varepsilon$  spaces are the relevant symmetric spaces for the geometric realization of the  $\Theta$ -hypergeometric transform, and include the non-compactly causal (NCC) symmetric spaces as special case. See Section 2 for more information.

The Paley–Wiener theorem for the  $\Theta$ -hypergeometric transform, which is the main result of this paper, is given by the following theorem.

**Theorem 1.3** (Paley–Wiener theorem). *Let  $\Theta \subset \Pi$  and let  $m \in \mathcal{M}^+$  be an even multiplicity function. Suppose Condition A is satisfied. Let  $C$  be a compact, convex and  $W_\Theta$ -invariant subset of  $\mathfrak{a}_\Theta$ . Then the  $\Theta$ -hypergeometric transform  $\mathcal{F}_\Theta(m)$  is a bijection of  $C_c^\infty(C)^{W_\Theta}$  onto  $\text{PW}_\Theta(m; C)$ .*

It is important to remark that Condition A only plays a role in the proof that the  $\Theta$ -hypergeometric transform is surjective. The injectivity will be proven without any restriction on the pair  $(m, \Theta)$ . See Theorem 8.1.

In the case  $m = 0$ , Theorem 1.3 reduces to the classical Paley–Wiener theorem for the Euclidean spherical transform. For  $\Theta = \Pi$ , it is a slight generalization of the Paley–Wiener theorem for the Odpm transform as stated in Theorems 8.6 and 9.13(4) in [28], where it has been proven for  $W$ -invariant convex sets of  $\mathfrak{a}$  of the form  $\text{conv}(W(H))$  with  $H \in \mathfrak{a}$ . Indeed in this case the Theorem holds without assuming that the multiplicities must be even. For  $\Theta = \Pi$  and  $m$  geometric, it reduces to the celebrated theorem of Helgason–Gangolli–Rosenberg (see, e.g., [13], Chapter IV, Theorem 7.1, or [8], Theorem 6.6.8). An elementary proof of Theorem 1.3 for  $\Theta = \Pi$  and  $m$  even is given on p. 32.

If  $(\mathfrak{a}, \Sigma, m)$  corresponds to a NCC symmetric space  $G/H$  and  $\Pi = \Theta_0$ , then Theorem 1.3 yields a Paley–Wiener theorem for the spherical Laplace transform on  $G/H$ . For a very special exhausting family of compact convex subsets in the maximal cone  $C_{\max} := \mathfrak{a}_{\Pi_0}$ , a Paley–Wiener type theorem in this context was also proven in [1]. However, our description of the Paley–Wiener space in Definition 1.2 is more explicit, and very close to the Euclidean one, since the exponential growth condition is only modified by multiplication by the polynomial  $e_\Theta^-$  giving the location of the possible singularities. The simplification of the Paley–Wiener space in the complex NCC case was also not remarked before. Finally, the proof presented in [1] depends heavily on the causal structure of the symmetric space, whereas our proof here applies to a much more general context, and is based only on the root structure and the fact that the multiplicities are even.

More precisely, the proof of Theorem 1.3 is based on a reduction to the Paley–Wiener theorem for the Euclidean Fourier transform. The tools allowing this reductions are  $W$ -invariant differential operators  $D_m$  on  $A$  linking the  $\Theta$ -hypergeometric functions for even multiplicities to the exponential functions. These differential operators are essentially special cases of the shift operators of Odpm [25]. Shift operators are differential operators with smooth coefficients on the positive Weyl chamber  $A^+$  but generally singular on the “walls” of  $A^+$ . The crucial observation is that the modified shift operators  $D_m$  we are using have smooth coefficients on all of  $A$ . In fact its coefficients are even holomorphic on a certain torus  $A_{\mathbb{C}}$  complexifying  $A$  (see Theorem 4.10 and Corollary 4.14).

An important intermediate step in the proof of the Paley–Wiener theorem is a new inversion formula for the  $\Theta$ -hypergeometric transform in the even multiplicity case. The formula resembles the inversion formula for the Euclidean Fourier transform, with only a correction due to the differential operator  $D_m$ . See Theorem 13.11 and Corollary 13.11.

The study of the modified shift operators  $D_m$  gives as a parallel result explicit formulas for the  $\Theta$ -hypergeometric functions corresponding to even multiplicities. These formulas

are of independent interest. They yield as special instances new explicit formulas for the spherical functions on Riemannian and NCC symmetric spaces with even multiplicities. See Theorem 5.1, Corollary 5.2, and Examples 5.3 and 5.4. We determine two types of formulas for the  $\Theta$ -hypergeometric functions. The first type involves the differential operator  $D_m$  applied to a sum over the Weyl group  $W_\Theta$  of exponential functions. This type is used in the proof of the Paley–Wiener theorem. The second type presents a differential operator applied to an alternating sum of exponential functions over the Weyl group  $W_\Theta$ . The formulas of the second type resemble the classical formulas by Harish-Chandra for the spherical functions on Riemannian symmetric spaces of noncompact type  $G/K$  with  $G$  complex. The second type is obtained from the first one by splitting  $D_m$  as a composition of a differential operator and the shift operator of constant shift 2.

The proof that the operator  $D_m$  have nonsingular coefficients depends on the behavior of the Harish-Chandra series and their derivatives in the spectral parameters on the walls of the positive Weyl chamber. This behavior can be deduced from new estimates, which hold for arbitrary positive multiplicity functions and which we have collected in Appendix A. The fact that the above mentioned splitting of  $D_m$  returns a differential operator is stated in Corollary 4.16, which is proven in Appendix B.

## 2. Preliminaries

Let  $\mathfrak{a}$  be an  $l$ -dimensional real Euclidean vector space with inner product  $\langle \cdot, \cdot \rangle$ . For every  $\alpha$  in the dual space  $\mathfrak{a}^*$  of  $\mathfrak{a}$ , let  $A_\alpha \in \mathfrak{a}$  be determined by  $\alpha(H) = \langle H, A_\alpha \rangle$  for all  $H \in \mathfrak{a}$ . Then for  $\alpha \neq 0$  the vector  $H_\alpha := 2A_\alpha / \langle A_\alpha, A_\alpha \rangle$  satisfies  $\alpha(H_\alpha) = 2$ . The assignment  $\langle \alpha, \beta \rangle := \langle A_\alpha, A_\beta \rangle$  defines an inner product in  $\mathfrak{a}^*$ . Let  $\Sigma$  be a reduced root system in  $\mathfrak{a}^*$  with associated Weyl group  $W$ . For every  $\alpha \in \Sigma$ , we denote by  $r_\alpha$  the reflection in  $\mathfrak{a}^*$  defined by  $r_\alpha(\lambda) := \lambda - \lambda(H_\alpha)\alpha$  for all  $\lambda \in \mathfrak{a}^*$ .

Let  $\Sigma^+$  be a choice of positive roots in  $\Sigma$  and  $\Pi = \{\alpha_1, \dots, \alpha_l\}$  the system of simple roots associated with  $\Sigma^+$ . The positive Weyl chamber  $\mathfrak{a}^+$  consists of the elements  $H \in \mathfrak{a}$  for which  $\alpha(H) > 0$  for all  $\alpha \in \Sigma^+$ .

The complexification  $\mathfrak{a}_\mathbb{C} := \mathfrak{a} \otimes_\mathbb{R} \mathbb{C}$  of  $\mathfrak{a}$  can be viewed as the Lie algebra of the complex torus  $A_\mathbb{C} := \mathfrak{a}_\mathbb{C} / \mathbb{Z}\{i\pi H_\alpha : \alpha \in \Sigma\}$ . The exponential map  $\exp : \mathfrak{a}_\mathbb{C} \rightarrow A_\mathbb{C}$  is the canonical projection of  $\mathfrak{a}_\mathbb{C}$  onto  $A_\mathbb{C}$ . Its multi-valued inverse is denoted by  $\log$ . The split real form  $A := \exp \mathfrak{a}$  of  $A_\mathbb{C}$  is an Abelian subgroup of  $A_\mathbb{C}$  with Lie algebra  $\mathfrak{a}$  and  $\exp : \mathfrak{a} \rightarrow A$  is a diffeomorphism. Set  $A^+ := \exp \mathfrak{a}^+$ . The polar decomposition of  $A_\mathbb{C}$  is  $A_\mathbb{C} = AT$ , where  $T := \exp(i\mathfrak{a})$  is a compact torus with Lie algebra  $i\mathfrak{a}$ . Let  $\mathfrak{a}_\mathbb{C}^*$  be the space of all  $\mathbb{C}$ -valued linear functionals on  $\mathfrak{a}$ . The action of  $W$  extends to  $\mathfrak{a}$  by duality, and then to  $\mathfrak{a}_\mathbb{C}^*$  and  $\mathfrak{a}_\mathbb{C}$  by  $\mathbb{C}$ -linearity, and to  $A_\mathbb{C}$  and  $A$  by the exponential map. Moreover,  $W$  acts on functions  $f$  on any of these spaces by  $(wf)(x) := f(w^{-1}x)$ ,  $w \in W$ . The  $\mathbb{C}$ -bilinear extension to  $\mathfrak{a}_\mathbb{C}^*$  and  $\mathfrak{a}_\mathbb{C}$  of the inner products  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{a}^*$  and  $\mathfrak{a}$  will also be denoted by  $\langle \cdot, \cdot \rangle$ .

A multiplicity function on  $\Sigma$  is a  $W$ -invariant function  $m : \Sigma \rightarrow \mathbb{C}$ . Setting  $m_\alpha := m(\alpha)$ , we therefore have  $m_{w\alpha} = m_\alpha$  for all  $\alpha \in \Sigma$  and  $w \in W$ . The set  $\mathcal{M}$  of all multiplicity functions on  $\Sigma$  is a subspace of the finite-dimensional  $\mathbb{C}$ -vector space  $\mathbb{C}^\Sigma$ . Given  $k \in \mathbb{C}$ , the multiplicity function with  $(km)_\alpha = km_\alpha$  for all  $\alpha \in \Sigma$  is denoted by  $km$ . The multiplicity function with constant value  $m_\alpha = k$  for all  $\alpha \in \Sigma$  is denoted by  $k$ . The

multiplicity function  $m$  is said to be positive (respectively, even) if  $m_\alpha \geq 0$  (respectively,  $m_\alpha \in 2\mathbb{Z}$ ) for all  $\alpha \in \Sigma$ . The set of positive multiplicity functions is denoted by  $\mathcal{M}^+$ . Finally, a multiplicity function  $m$  is said to be *geometric* if there is a Riemannian symmetric space of noncompact type  $G/K$  with restricted root system  $\Sigma$  such that  $m_\alpha$  is the multiplicity of the root  $\alpha$  for all  $\alpha \in \Sigma$ . Otherwise,  $m$  is said to be *non-geometric*.<sup>2</sup>

In the geometric case, even multiplicities correspond to Riemannian symmetric spaces  $G/K$  with the property that all Cartan subalgebras in the Lie algebra  $\mathfrak{g}$  of  $G$  are conjugate under the adjoint group of  $\mathfrak{g}$  (see [12], p. 429). The simplest examples correspond to spaces where  $G$  is complex, for which all multiplicities are equal to 2. Among the pseudo-Riemannian symmetric spaces, the material developed in this paper is particularly relevant for the study of spherical functions on the  $K_\varepsilon$ -spaces and the so-called non-compactly causal (NCC) symmetric spaces (see, e.g., [15]). We shall assume that  $G$  is simple. Let  $\mathfrak{k}$  be the Lie algebra of  $K$  and denote by  $\theta$  the corresponding Cartan involution. Denote by  $\mathfrak{p}$  the  $(-1)$ -eigenspace of  $\theta$ . We assume that  $\mathfrak{a}$  is a maximal Abelian subspace of  $\mathfrak{p}$ . Set  $\mathfrak{m} = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})$  and  $\mathfrak{g}^\alpha = \{X \in \mathfrak{g}^\alpha : [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{a}\}$ . A map  $\varepsilon : \Sigma \rightarrow \{1, -1\}$  is a *signature* if  $\varepsilon(\alpha + \beta) = \varepsilon(\alpha)\varepsilon(\beta)$  whenever  $\alpha, \beta, \alpha + \beta \in \Sigma$  and  $\varepsilon(\alpha) = \varepsilon(-\alpha)$  for all  $\alpha \in \Sigma$ . Let  $\varepsilon$  be a signature. Define an involution  $\theta_\varepsilon$  on  $\mathfrak{g}$  by:

$$\theta_\varepsilon|_{\mathfrak{m} \oplus \mathfrak{a}} = \theta|_{\mathfrak{m} \oplus \mathfrak{a}}$$

and for each  $\alpha \in \Sigma$  by:

$$\theta_\varepsilon|_{\mathfrak{g}^\alpha} = \varepsilon(\alpha)\theta|_{\mathfrak{g}^\alpha}.$$

Denote also by  $\theta_\varepsilon$  the corresponding involution on  $G$ , and let  $H = \{g \in G : \theta_\varepsilon(g) = g\}$ . Then  $G/H$  is called a  $K_\varepsilon$ -space. If there exists a  $X_0 \in \mathfrak{a}$  such that  $\alpha(X_0) \in \{0, 1, -1\}$  for all  $\alpha \in \Sigma$  and the signature  $\varepsilon$  is given by:

$$\varepsilon(\alpha) = (-1)^{\alpha(X_0)}, \quad (6)$$

then  $G/H$  is a non-compactly causal symmetric space. We notice that the root system and the multiplicity function corresponding to the symmetric space  $G/H$  is the same as that of  $G/K$ . The infinitesimal classification of the  $K_\varepsilon$ -spaces  $G/H$  with simple  $G$  can be found in the appendix of [30]. The root multiplicities can be read off from the list. We list all the  $K_\varepsilon$  symmetric pairs with even multiplicities in Appendix C. An interesting fact, which however will not be used in the present paper, is that all multiplicities of a  $K_\varepsilon$ -space with even multiplicities are equal.

For  $\alpha \in \Sigma$  and  $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$  we set:

$$\lambda_\alpha := \frac{\lambda(H_\alpha)}{2} = \frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle}. \quad (7)$$

<sup>2</sup> We adopt the multiplicity notation commonly used in the theory of symmetric spaces. It differs from the notation employed by Heckman and Opdam in the following ways. The root system  $R$  used by Heckman and Opdam is related to our root system  $\Sigma$  by the relation  $R = \{2\alpha : \alpha \in \Sigma\}$ ; the multiplicity function  $k$  in Heckman and Opdam's work is related to our  $m$  by  $k_{2\alpha} = m_\alpha/2$ .

The *restricted weight lattice* of  $\Sigma$  is the set  $P$  of all  $\lambda \in \mathfrak{a}^*$  with  $\lambda_\alpha \in \mathbb{Z}$  for all  $\alpha \in \Sigma$ . Notice that  $2\alpha \in P$  for all  $\alpha \in \Sigma$ . The elements in  $\mathfrak{a}_\mathbb{C}^* \setminus P$  are said to be *generic*.

If  $\lambda \in P$ , then the exponential  $e^\lambda$  defined by  $e^\lambda(a) := e^{\lambda(\log a)}$  is single valued on  $A_\mathbb{C}$ . The  $\mathbb{C}$ -linear span of the  $e^\lambda$ ,  $\lambda \in P$ , is the ring of regular functions  $\mathbb{C}[A_\mathbb{C}]$  on the affine algebraic variety  $A_\mathbb{C}$ . The lattice  $P$  is  $W$ -invariant, and the Weyl group acts on  $\mathbb{C}[A_\mathbb{C}]$  according to  $w(e^\lambda) := e^{w\lambda}$  for all  $w \in W$ . The set

$$A_\mathbb{C}^{\text{reg}} := \{h \in A_\mathbb{C} : e^{2\alpha(\log h)} \neq 1 \text{ for all } \alpha \in \Sigma\}$$

consists of the regular points of  $A_\mathbb{C}$  for the action of  $W$ . Notice that  $A^+$  is a subset of  $A_\mathbb{C}^{\text{reg}}$ . The algebra  $\mathbb{C}[A_\mathbb{C}^{\text{reg}}]$  of regular functions on  $A_\mathbb{C}^{\text{reg}}$  is the subalgebra of the quotient field  $\mathbb{C}(A_\mathbb{C})$  of  $\mathbb{C}[A_\mathbb{C}]$  generated by  $\mathbb{C}[A_\mathbb{C}]$  and  $1/(1 - e^{-2\alpha})$ ,  $\alpha \in \Sigma^+$ . Its  $W$ -invariant elements form the subalgebra  $\mathbb{C}[A_\mathbb{C}^{\text{reg}}]^W$ .

Given a multiplicity function  $m \in \mathcal{M}$  on  $\Sigma$ , we define:

$$\rho(m) := \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha, \quad (8)$$

$$\Delta(m) := \prod_{\alpha \in \Sigma^+} (e^\alpha - e^{-\alpha})^{m_\alpha} = e^{2\rho(m)} \prod_{\alpha \in \Sigma^+} (1 - e^{-2\alpha})^{m_\alpha}. \quad (9)$$

In particular,  $\Delta := \Delta(1)$  is the *Weyl denominator*.

**Lemma 2.1.** *Let  $\Sigma$  be a reduced root system and let  $m \in \mathcal{M}^+$  be an even multiplicity function. Then  $\rho \in P$ . Consequently,  $\Delta(m/2) \in \mathbb{C}[A_\mathbb{C}]$ . Furthermore,  $\Delta(m) \in \mathbb{C}[A_\mathbb{C}]^W$ .*

**Proof.** If  $\alpha \in \Pi$  is a simple root, then  $\rho_\alpha = m_\alpha/2 \in \mathbb{Z}$ . For general  $\alpha \in \Sigma$  there are  $\beta \in \Pi$  and  $w \in W$  so that  $\alpha = w\beta$ . The property that  $\rho_\alpha \in \mathbb{Z}$  follows then by induction on the length of  $w$ . This proves  $\rho \in P$ . Since  $-2\alpha \in P$  and  $m_\alpha/2 \in \mathbb{Z}$ , we also obtain:

$$\Delta(m/2) = e^{\rho(m)} \prod_{\alpha \in \Sigma^+} (1 - e^{-2\alpha})^{m_\alpha/2} \in \mathbb{C}[A_\mathbb{C}].$$

Let  $c = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha$ . Then

$$\Delta(m) = (-1)^c \prod_{\alpha \in \Sigma^+} (e^\alpha - e^{-\alpha})^{m_\alpha/2} \prod_{\alpha \in \Sigma^+} (e^{-\alpha} - e^\alpha)^{m_\alpha/2} = (-1)^c \prod_{\alpha \in \Sigma} (e^\alpha - e^{-\alpha})^{m_\alpha/2},$$

which proves that  $\Delta(m) \in \mathbb{C}[A_\mathbb{C}]^W$ .  $\square$

Let  $S(\mathfrak{a}_\mathbb{C})$  denote the symmetric algebra over  $\mathfrak{a}_\mathbb{C}$  considered as the space of polynomial functions on  $\mathfrak{a}_\mathbb{C}^*$ , and let  $S(\mathfrak{a}_\mathbb{C})^W$  be the subalgebra of  $W$ -invariant elements. For  $p \in S(\mathfrak{a}_\mathbb{C})$  write  $\partial(p)$  for the corresponding constant-coefficient differential operator on  $A_\mathbb{C}$  (or on



$\mathfrak{a}_{\mathbb{C}}$ ). Let  $\mathbb{D}(A_{\mathbb{C}}^{\text{reg}}) := \mathbb{C}[A_{\mathbb{C}}^{\text{reg}}] \otimes S(\mathfrak{a}_{\mathbb{C}})$  denote the algebra of differential operators on  $A_{\mathbb{C}}$  with coefficients in  $\mathbb{C}[A_{\mathbb{C}}^{\text{reg}}]$ . The Weyl group  $W$  acts on  $\mathbb{D}(A_{\mathbb{C}}^{\text{reg}})$  according to

$$w(\phi \otimes \partial(p)) := w\phi \otimes \partial(wp).$$

Let  $\mathbb{D}(A_{\mathbb{C}}^{\text{reg}})^W$  denote the subspace of  $W$ -invariant elements in  $\mathbb{D}(A_{\mathbb{C}}^{\text{reg}})$ . The set  $\mathbb{D}(A_{\mathbb{C}}^{\text{reg}}) \otimes \mathbb{C}[W]$  of differential reflection operators on  $A_{\mathbb{C}}^{\text{reg}}$  can be endowed with the structure of an associative algebra with respect to the product

$$(D_1 \otimes w_1) \cdot (D_2 \otimes w_2) = D_1 w_1(D_2) \otimes w_1 w_2,$$

where the action of  $W$  on differential operators is defined by  $(wD)(wf) := w(Df)$  for every sufficiently differentiable function  $f$ . It is also a left  $\mathbb{C}[A_{\mathbb{C}}^{\text{reg}}]$ -module. Considering  $D \in \mathbb{D}(A_{\mathbb{C}}^{\text{reg}})$  as element of  $\mathbb{D}(A_{\mathbb{C}}^{\text{reg}}) \otimes \mathbb{C}[W]$ , we shall usually write  $D$  instead of  $D \otimes 1$ . The differential-reflection operators act on functions  $f$  on  $A_{\mathbb{C}}^{\text{reg}}$  according to  $(D \otimes w)f := D(wf)$ .

Define a linear map  $\Upsilon : \mathbb{D}(A_{\mathbb{C}}^{\text{reg}}) \otimes \mathbb{C}[W] \rightarrow \mathbb{D}(A_{\mathbb{C}}^{\text{reg}})$  by:

$$\Upsilon\left(\sum_j D_j \otimes w_j\right) := \sum_j D_j.$$

Then  $\Upsilon(Q)f = Qf$  for all  $Q \in \mathbb{D}(A_{\mathbb{C}}^{\text{reg}}) \otimes \mathbb{C}[W]$  and all  $W$ -invariant functions  $f$  on  $A_{\mathbb{C}}^{\text{reg}}$ .

**Definition 2.2** [4]. Let  $m \in \mathcal{M}$  and  $H \in \mathfrak{a}_{\mathbb{C}}$ . The Dunkl–Cherednik operator  $T(m; H) \in \mathbb{D}(A_{\mathbb{C}}^{\text{reg}}) \otimes \mathbb{C}[W]$  is defined by:

$$T(m; H) := \partial(H) - \rho(m)(H) + \sum_{\alpha \in \Sigma^+} m_{\alpha} \alpha(H) (1 - e^{-2\alpha})^{-1} \otimes (1 - r_{\alpha}).$$

The Dunkl–Cherednik operators map  $\mathbb{C}[A_{\mathbb{C}}]$  into itself, but they can also be considered as operators acting on other function spaces, for instance, on the spaces of smooth and compactly supported smooth functions on  $A$  and  $\mathfrak{a}$ . Indeed, as can be seen from the Taylor formula, the term  $1 - r_{\alpha}$  cancels the apparent singularity on  $A$  and  $\mathfrak{a}$  arising from the denominator  $1 - e^{-2\alpha}$ .

The Dunkl–Cherednik operators  $\{T(m; H) : H \in \mathfrak{a}_{\mathbb{C}}\}$  form a commuting family of differential-reflection operators in  $\mathbb{D}(A_{\mathbb{C}}^{\text{reg}}) \otimes \mathbb{C}[W]$  (cf. [28], Section 2). Therefore the map  $H \mapsto T(m; H)$  on  $\mathfrak{a}_{\mathbb{C}}$  extends uniquely to an algebra homomorphism of  $S(\mathfrak{a}_{\mathbb{C}})$  into  $\mathbb{D}(A_{\mathbb{C}}^{\text{reg}}) \otimes \mathbb{C}[W]$ . For  $p \in S(\mathfrak{a}_{\mathbb{C}})$  we set  $D(m; p) := \Upsilon(T(m; p))$ . Then  $\Upsilon$  establishes an algebra homomorphism of  $\{T(m; p) : p \in S(\mathfrak{a}_{\mathbb{C}})^W\}$  into  $\mathbb{D}(A_{\mathbb{C}}^{\text{reg}})^W$  (see [11], Lemma 1.2.2). The differential operator  $D(m; p)$  is the unique element of  $\mathbb{D}(A_{\mathbb{C}}^{\text{reg}})^W$  coinciding on  $\mathbb{C}[A_{\mathbb{C}}^{\text{reg}}]^W$  with the restriction of  $T(m; p)$ . The algebra

$$\mathbb{D}(\mathfrak{a}, \Sigma, m) := \{D(m; p) : p \in S(\mathfrak{a}_{\mathbb{C}})^W\} \quad (10)$$

is a commutative subalgebra of  $\mathbb{D}(A_{\mathbb{C}}^{\text{reg}})^W$ . It is called the algebra of hypergeometric differential operators associated with the data  $(\mathfrak{a}, \Sigma, m)$ .

Let  $L_A$  denote the Laplace operator on  $A$  and let  $p_L \in S(\mathfrak{a}_{\mathbb{C}})^W$  be the polynomial defined by  $p_L(\lambda) := \langle \lambda, \lambda \rangle$  for  $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ . Then

$$ML(m) := D(m; p_L) = L(m) + \langle \rho(m), \rho(m) \rangle, \quad (11)$$

where

$$L(m) := L_A + \sum_{\alpha \in \Sigma^+} m_{\alpha} \frac{1 + e^{-2\alpha}}{1 - e^{-2\alpha}} \partial(A_{\alpha}), \quad (12)$$

generalizes to arbitrary multiplicity functions  $m$  the radial component on  $A^+$  of the Laplace operator on a Riemannian symmetric space  $G/K$  of noncompact type. The algebra  $\mathbb{D}(\mathfrak{a}, \Sigma, m)$  coincides with the commutant  $\{Q \in \mathbb{D}(A_{\mathbb{C}}^{\text{reg}})^W : L(m)Q = QL(m)\}$  of  $L(m)$  in  $\mathbb{D}(A_{\mathbb{C}}^{\text{reg}})^W$ . It is therefore the analog, for arbitrary multiplicity functions, of the commutative algebra of the radial parts on  $A$  of the invariant differential operators on a Riemannian symmetric space of noncompact type. The map  $\gamma(m) : \mathbb{D}(\mathfrak{a}, \Sigma, m) \rightarrow S(\mathfrak{a}_{\mathbb{C}})^W$  defined by:

$$\gamma(m)(D(m; p))(\lambda) := p(\lambda) \quad (13)$$

is an algebra isomorphism, called the Harish-Chandra homomorphism (see [11], Theorem 1.3.12 and Remark 1.3.14). Chevalley's theorem implies that  $\mathbb{D}(\mathfrak{a}, \Sigma, m)$  is generated by  $l$  ( $= \dim \mathfrak{a}$ ) elements.

Let  $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$  be fixed. The system of differential equations

$$D(m; p)\varphi = p(\lambda)\varphi, \quad p \in S(\mathfrak{a}_{\mathbb{C}})^W, \quad (14)$$

is called the hypergeometric system of differential equations with spectral parameter  $\lambda$  associated with the data  $(\mathfrak{a}, \Sigma, m)$ . For geometric multiplicities, the hypergeometric system (14) agrees with the system of differential equations on  $A$  defining Harish-Chandra's spherical function of spectral parameter  $\lambda$ .

In the following we denote by  $\mathbb{N}$  the set of positive integers and set  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ .

### 3. $\Theta$ -hypergeometric functions

As in the theory of spherical functions on Riemannian symmetric spaces of noncompact type, Heckman and Opdam [10] looked for solutions of the hypergeometric system (14) with spectral parameter  $\lambda$  which are of the form

$$\Phi(m; \lambda, a) = e^{(\lambda - \rho)(\log a)} \sum_{\mu \in 2\Lambda} \Gamma_{\mu}(m; \lambda) e^{-\mu(\log a)} \quad (15)$$

for all  $a$  in some tubular neighborhood of  $A^+$  in  $A_{\mathbb{C}}$ . Here  $\Lambda := \{\sum_{j=1}^l n_j \alpha_j : n_j \in \mathbb{N}_0\}$  is the positive semigroup generated by the fundamental system of simple roots  $\Pi := \{\alpha_1, \dots, \alpha_l\}$  in  $\Sigma^+$ . For  $\mu \in 2\Lambda \setminus \{0\}$ , the coefficients  $\Gamma_{\mu}(m; \lambda)$  are rational functions of  $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$  determined from the recursion relations:

$$\langle \mu, \mu - 2\lambda \rangle \Gamma_{\mu}(m; \lambda) = 2 \sum_{\alpha \in \Sigma^+} m_{\alpha} \sum_{\substack{k \in \mathbb{N} \\ \mu - 2k\alpha \in \Lambda}} \Gamma_{\mu - 2k\alpha}(m; \lambda) \langle \mu + \rho - 2k\alpha - \lambda, \alpha \rangle, \quad (16)$$

which are derived by formally inserting the series for  $\Phi$  into the differential equation of (14) corresponding to  $p = p_L$ . With the initial condition  $\Gamma_0(m; \lambda) = 1$ , the relations (16) admit unique solutions  $\Gamma_{\mu}(m; \lambda)$  provided  $\langle \mu, \mu - 2\lambda \rangle \neq 0$  for all  $\mu \in 2\Lambda \setminus \{0\}$ . The functions  $\Phi(m; \lambda, a)$  are commonly known as the Harish-Chandra series.

**Theorem 3.1** (a) ([26], Corollary 2.3; see also [27], Lemma 2.1). *There is a connected and simply connected open subset  $U$  of  $T$  containing the identity element  $e$  such that  $\Phi(m; \lambda, a)$  is a meromorphic functions of  $(m, \lambda, a) \in \mathcal{M} \times \mathfrak{a}_{\mathbb{C}}^* \times A^+U$  with at most simple poles along hyperplanes of the form  $\mathcal{M} \times \mathcal{H}_{n,\alpha} \times A^+U$ , where*

$$\mathcal{H}_{n,\alpha} := \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* : \lambda_{\alpha} = n\}$$

*is a complex hyperplane in  $\mathfrak{a}_{\mathbb{C}}^*$  corresponding to some  $\alpha \in \Sigma^+$  and  $n \in \mathbb{N}$ .*

(b) [11, Corollary 4.2.6]. *For  $\lambda \in \mathfrak{a}_{\mathbb{C}}^* \setminus P$  the set  $\{\Phi(m; w\lambda, a) : w \in W\}$  is a basis for the  $C^\infty$  solution space of (14) on  $A^+U$ .*

The set  $U$  in Theorem 3.1 is chosen so that the function  $\log$  is single valued on it, i.e., so that  $e^{(\lambda - \rho(m))(\log a)}$  is holomorphic on  $U$  for all  $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ . Observe that there is a neighborhood  $V$  of  $\lambda = 0$  in  $\mathfrak{a}_{\mathbb{C}}^*$  such that  $\Phi(m; \lambda, a)$  is a holomorphic function of  $(m, \lambda, a) \in \mathcal{M} \times V \times A^+U$ .

Let  $\Theta \subset \Pi$  be an arbitrary set of positive simple roots, and let  $\langle \Theta \rangle = \Sigma \cap \text{span } \Theta$  be the subsystem of  $\Sigma$  generated by  $\Theta$ . Let  $\langle \Theta \rangle^+ := \Sigma^+ \cap \langle \Theta \rangle$ . Then the Weyl group  $W_{\Theta}$  of  $\langle \Theta \rangle$  is the subgroup of  $W$  generated by the reflections  $r_{\alpha}$  with  $\alpha \in \Theta$ .

Given an even multiplicity function  $m \in \mathcal{M}^+$ , we define:

$$\begin{aligned} c_{\Theta}^+(m; \lambda) &:= \prod_{\alpha \in \langle \Theta \rangle^+} \prod_{k=0}^{m_{\alpha}/2-1} \frac{1}{\lambda_{\alpha} + k}, \\ c_{\Theta}^-(m; \lambda) &:= \prod_{\alpha \in \Sigma^+ \setminus \langle \Theta \rangle^+} \prod_{k=-m_{\alpha}/2+1}^0 \frac{1}{-\lambda_{\alpha} + k} \\ &= (-1)^{d(\Theta, m)} \prod_{\alpha \in \Sigma^+ \setminus \langle \Theta \rangle^+} \prod_{k=0}^{m_{\alpha}/2-1} \frac{1}{\lambda_{\alpha} + k}, \end{aligned}$$

where

$$d(\Theta, m) := \frac{1}{2} \sum_{\alpha \in \Sigma^+ \setminus \langle \Theta \rangle^+} m_\alpha. \quad (17)$$

**Definition 3.2.** Suppose  $\lambda \in \mathfrak{a}_\mathbb{C}^* \setminus P$ . Then the  $\Theta$ -hypergeometric function of spectral parameter  $\lambda$  is the  $W_\Theta$ -invariant solution of (14) defined on  $A^+U$  by:

$$\varphi_\Theta(m; \lambda, a) := c_\Theta^-(m; \lambda) \sum_{w \in W_\Theta} c_\Theta^+(m; w\lambda) \Phi(m; w\lambda, a). \quad (18)$$

To underline their link with the harmonic analysis on symmetric spaces, the  $\Theta$ -hypergeometric functions corresponding to geometric multiplicities will be called  $\Theta$ -spherical functions.

**Remark 3.3.** The functions  $c_\Theta^+$  and  $c_\Theta^-$  are respectively the analog for  $\langle \Theta \rangle$  of Harish-Chandra  $c$ -function for Riemannian symmetric spaces and of the  $c$ -function of [20] for NCC spaces. We refer the reader to [22,31] and [32] for their general definition and for further information on  $\Theta$ -hypergeometric functions corresponding to arbitrary multiplicity functions.

**Example 3.4.** When  $\Theta = \Pi$ , the function  $\varphi_\Pi(m; \lambda, a)/c_\Pi^+(m; \rho(m))$  coincides with the hypergeometric function of spectral parameter  $\lambda$  as defined by Heckman and Opdam [10]. For geometric multiplicity functions, it therefore agrees with the restriction of Harish-Chandra's spherical function of spectral parameter  $\lambda$  to a maximal flat subspace  $A$  of the corresponding Riemannian symmetric space  $G/K$ .

**Example 3.5.** Suppose  $\Sigma$  is the restricted root system of a NCC symmetric space  $G/H$ . If  $\Theta = \Pi_0$  is the set of positive compact simple roots and  $m$  is geometric, then the function  $\varphi_{\Pi_0}(m; \lambda, a)/c_{\Pi_0}^+(m; \rho(m))c_{\Pi_0}^-(m; \rho(m))$  coincides with the spherical function on  $G/H$  with spectral parameter  $\lambda$  as defined by [6] (see [21]).

Set

$$\mathfrak{a}_\Theta := (W_\Theta(\overline{\mathfrak{a}^+}))^0 = \{H \in \mathfrak{a} : \alpha(H) > 0 \text{ for all } \alpha \in \Sigma^+ \setminus \langle \Theta \rangle^+\}, \quad (19)$$

$$A_\Theta := (W_\Theta(\overline{A^+}))^0 = \exp(\mathfrak{a}_\Theta). \quad (20)$$

The regularity properties of the  $\Theta$ -hypergeometric functions for even multiplicity functions are collected in the following theorem.

**Theorem 3.6** [23, Theorem 10]. *Define:*

$$e_\Theta^-(m; \lambda) := \prod_{\alpha \in \Sigma^+ \setminus \langle \Theta \rangle^+} \prod_{k=-m_\alpha/2+1}^{m_\alpha/2-1} (\lambda_\alpha - k). \quad (21)$$

Then there is a  $W_\Theta$ -invariant tubular neighborhood  $U_\Theta$  in  $A_\mathbb{C}$  of  $A_\Theta$  such that the function

$$e_\Theta^-(m; \lambda) \varphi_\Theta(m; \lambda, h)$$

extends as a  $W_\Theta$ -invariant holomorphic function of  $(\lambda, h) \in \mathfrak{a}_\mathbb{C}^* \times U_\Theta$ .

Estimates for the  $\Theta$ -hypergeometric functions and their derivatives for  $\Theta = \Pi$  and  $m \in \mathcal{M}^+$  arbitrary have been proven by Opdam.

**Lemma 3.7** (cf. [28], Theorem 3.15, Proposition 6.1 and Corollary 6.2). *Let  $m \in \mathcal{M}^+$  be a fixed multiplicity function. For  $a \in A_\mathbb{C}$  write  $\log a = a_\mathbb{R} + ia_\mathbb{I}$  with  $a_\mathbb{R}, a_\mathbb{I} \in \mathfrak{a}$ .*

(a) *For all  $\lambda \in \mathfrak{a}_\mathbb{C}^*$  and all  $a \in A_\mathbb{C}$  satisfying  $|\alpha(a_\mathbb{I})| < \pi/2$  for every  $\alpha \in \Sigma$ ,*

$$\begin{aligned} |\varphi_\Pi(m; \lambda, a)| &\leq |W|^{1/2} c_\Pi^+(m; \rho(m)) \\ &\quad \times e^{-\min_{w \in W} \operatorname{Im} w\lambda(a_\mathbb{I}) + \max_{w \in W} w\rho(a_\mathbb{I}) + \max_{w \in W} \operatorname{Re} w\lambda(a_\mathbb{R})}. \end{aligned}$$

*In particular, for all  $a \in A$  and  $\lambda \in \mathfrak{a}_\mathbb{C}^*$ ,*

$$|\varphi_\Pi(m; \lambda, a)| \leq |W|^{1/2} c_\Pi^+(m; \rho(m)) e^{\max_{w \in W} \operatorname{Re} w\lambda(\log a)}. \quad (22)$$

(b) *Let  $I = (i_1, \dots, i_l)$  be a multi-index and let  $\partial_a^I = \partial(H_1)^{i_1} \cdots \partial(H_l)^{i_l}$  be the corresponding partial differential operator associated with an orthonormal basis  $\{H_1, \dots, H_l\}$  of  $\mathfrak{a}$ . Then, for every  $\varepsilon > 0$ , there is a constant  $C_{I,\varepsilon} > 0$  such that for all  $a \in A$  and  $\lambda \in i\mathfrak{a}^*$  with  $|\lambda| > \varepsilon$ ,*

$$|\partial_a^I \varphi_\Pi(m; \lambda, a)| \leq C_{I,\varepsilon} |\lambda|^{|I|+l},$$

*where  $|I| := \sum_{k=1}^l i_k$ . In particular, if  $K \subset A$  is compact, then there is a constant  $C_{I,K} > 0$  so that for all  $a \in K$  and  $\lambda \in i\mathfrak{a}^*$ ,*

$$|\partial_a^I \varphi_\Pi(m; \lambda, a)| \leq C_{I,K} (1 + |\lambda|)^{|I|+l}. \quad (23)$$

The definition of the  $\Theta$ -hypergeometric functions and analytic continuation yield the following important functional relation.

**Lemma 3.8.** *Let  $m \in \mathcal{M}^+$  be even, and let  $d(\Theta, m)$  be as in (17). Then there is a  $W$ -invariant neighborhood of  $A_\Theta$  in  $A_\mathbb{C}$  on which the relation*

$$\varphi_\Pi(m; \lambda, a) = (-1)^{d(\Theta, m)} \sum_{w \in W_\Theta \setminus W} \varphi_\Theta(m; w\lambda, a),$$

*holds as equality of meromorphic functions on  $\mathfrak{a}_\mathbb{C}^*$ .*

We shall prove in Section 5 that, in the case of even multiplicity functions, explicit global formulas for the  $\Theta$ -hypergeometric functions can be obtained by means of Opdam's shift operators.

#### 4. Harish-Chandra series and Opdam's shift operators

Let  $\mathbb{C}_\Delta[A_\mathbb{C}] = \bigcup_{k \in \mathbb{Z}} \Delta^k \mathbb{C}[A_\mathbb{C}]$  denote the localization of  $\mathbb{C}[A_\mathbb{C}]$  along the Weyl denominator  $\Delta = \prod_{\alpha \in \Sigma^+} (e^\alpha - e^{-\alpha}) \in \mathbb{C}[A_\mathbb{C}]$ . Then  $\mathbb{C}[\mathcal{M}] \otimes \mathbb{C}_\Delta[A_\mathbb{C}] \otimes S(\mathfrak{a}_\mathbb{C})$  is the algebra of the differential operators on  $A_\mathbb{C}$  (or  $\mathfrak{a}_\mathbb{C}$ ) with coefficients in  $\mathbb{C}_\Delta[A_\mathbb{C}]$  depending also polynomially on the multiplicities. Recall from (11) the notation  $ML(m) := L(m) + \langle \rho(m), \rho(m) \rangle$ .

**Definition 4.1** ([25]; see also [11], Chapter 3). Let  $l \in \mathcal{M}$  be an even multiplicity function. A shift operator with shift  $l$  is a differential operator  $D(l) \in \mathbb{C}[\mathcal{M}] \otimes \mathbb{C}_\Delta[A_\mathbb{C}] \otimes S(\mathfrak{a}_\mathbb{C})$  satisfying the following properties:

- (i) For all  $m \in \mathcal{M}$ ,

$$D(l; m) \circ ML(m) = ML(m + l) \circ D(l; m).$$

- (ii) The differential operator  $D(l)$  admits on  $A^+$  an expansion of the form:

$$D(l) = \sum_{\mu \in 2\Lambda} e^{-\rho(l) - \mu} \partial(p_\mu) \quad (24)$$

with  $p_\mu \in \mathbb{C}[\mathcal{M}] \otimes S(\mathfrak{a}_\mathbb{C})$ .

**Remark 4.2.** In Definition 4.1(i) we have set  $D(l; m) := D(l)(m)$ , where  $m$  denotes the variable in  $\mathcal{M}$  from which  $D(l)$  depends polynomially. The asymptotics of  $D(l)$  in (ii) correspond to the power series expansion  $(e^\alpha - e^{-\alpha})^{-1} = \sum_{k=1}^{\infty} e^{-2k\alpha}$  on  $A^+$ . Moreover, the elements of  $\mathbb{C}[\mathcal{M}]$  act in (24) as constants, i.e.,  $\partial(q \otimes p) := q \partial(p)$  for  $q \in \mathbb{C}[\mathcal{M}]$  and  $p \in S(\mathfrak{a}_\mathbb{C})$ .

All shift operators turn out to be  $W$ -invariant. See, e.g., [11], Corollary 3.1.4. The name “shift operator of shift  $l$ ” reflects the property of  $D(l)$  of relating Harish-Chandra series corresponding to multiplicity functions differing by the shift  $l$ . This is the content of the next theorem.

In the following we denote by  $D^*$  the formal transpose of a differential operator  $D$  on  $A_\mathbb{C}^{\text{reg}}$  with respect to the Haar measure  $da$  on  $A$ :

$$\int_A (Df)g \, da = \int_A f(D^*g) \, da$$

for all smooth functions  $f, g$  on  $A^{\text{reg}} := A_{\mathbb{C}}^{\text{reg}} \cap A$  with at least one of  $f, g$  with compact support.

**Theorem 4.3** [25]. *Let  $l \in \mathcal{M}^+$  be an even multiplicity function.*

(a) *There exists a unique shift operator  $G_-(-l)$  of shift  $-l$  such that*

$$G_-(-l; m)\Phi(m; \lambda, a) = \frac{c_{\Pi}^+(m-l; \lambda)}{c_{\Pi}^+(m; \lambda)} \Phi(m-l; \lambda, a)$$

*for all  $(m, \lambda, a) \in \mathcal{M} \times (\mathfrak{a}_{\mathbb{C}}^* \setminus P) \times A^+$ .*

(b) *Let  $\Delta(m)$  be as in (9). Then*

$$G_+(l; m) := \Delta(-l-m) \circ G_-^*(-l; m+l) \circ \Delta(m) \quad (25)$$

*defines the unique shift operator  $G_+(l)$  of shift  $l$  such that*

$$G_+(l; m)\Phi(m; \lambda, a) = \frac{c_{\Pi}^+(m; -\lambda)}{c_{\Pi}^+(m+l; -\lambda)} \Phi(m+l; \lambda, a)$$

*for all  $(m, \lambda, a) \in \mathcal{M} \times (\mathfrak{a}_{\mathbb{C}}^* \setminus P) \times A^+$ .*

**Proof.** The existence of  $G_-(-l)$  and  $G_+(l)$  is proven in [11], Theorem 3.4.3 and Corollary 3.4. Their uniqueness depends on the injectivity of the Harish-Chandra mapping for shift operators (loc. cit., Proposition 3.1.6; see also Theorems 3.3.6 and 3.3.7, and the remark after the proof of Corollary 3.4.4).  $\square$

**Remark 4.4.** Obviously  $G_{\pm}(0; m) = \text{id}$  for all  $m \in \mathcal{M}$ .

The multiplicity  $m \equiv 0$  corresponds to the Euclidean case with  $\Phi(0; \lambda, a) = e^{\lambda(\log a)}$ . Hence Theorem 4.3 provides a link between Harish-Chandra functions for even multiplicities and exponential functions.

**Corollary 4.5.** *Let  $m \in \mathcal{M}$  be even. Then the differential operators in  $\mathbb{C}_{\Delta}[A_{\mathbb{C}}] \otimes S(\mathfrak{a}_{\mathbb{C}})$*

$$D_-(m) := G_-(-m; m), \quad (26)$$

$$D_+(m) := G_+(m; 0) = \Delta(-m) \circ G_-^*(-m; m) = \Delta(-m) \circ D_-^*(m) \quad (27)$$

*satisfy for all  $(\lambda, a) \in (\mathfrak{a}_{\mathbb{C}}^* \setminus P) \times A^+$ ,*

$$D_-(m)\Phi(m; \lambda, a) = \frac{1}{c_{\Pi}^+(m; \lambda)} e^{\lambda(\log a)}, \quad (28)$$

$$D_+(m)e^{\lambda(\log a)} = \frac{1}{c_{\Pi}^+(m; -\lambda)} \Phi(m; \lambda, a). \quad (29)$$

**Remark 4.6.** Suppose  $m \in \mathcal{M}$  is even. The uniqueness of the shift operators  $G_{\pm}$  implies for even  $l', l'' \in \mathcal{M}^+$ ,

$$G_-(-l' - l''; m) = G_-(-l'; m - l'') \circ G_-(-l''; m),$$

$$G_+(l' + l''; m) = G_+(l'; m + l'') \circ G_+(l''; m).$$

In particular, suppose  $m \in \mathcal{M}$  is even with  $m_{\alpha} \geq 2$  for all  $\alpha$ . Then

$$\begin{aligned} D_-(m) &:= G_-(-m; m) = G_-(-2; 2) \circ G_-(2 - m; m) \\ &= D_-(2) \circ G_-(-(m - 2); m), \end{aligned} \quad (30)$$

$$D_+(m) := G_+(m; 0) = G_+(m - 2; 2) \circ G_+(2; 0) = G_+(m - 2; 2) \circ D_+(2). \quad (31)$$

Furthermore, shift operators of even constant shift  $m \in \mathcal{M}$  can be obtained as composition of fundamental shift operators corresponding to shifts  $\pm 2$ . For instance, when  $m_{\alpha} = m \in 2\mathbb{N}$  for all  $\alpha \in \Sigma$ , we have:

$$G_-(-m; m) = G_-(-2; 2) \circ \cdots \circ G_-(-2; m - 2) \circ G_-(-2; m),$$

$$G_+(m; 0) = G_+(2; m - 2) \circ \cdots \circ G_+(2; 2) \circ G_+(2; 0).$$

Explicit formulas for  $D_-(2) = G_-(-2; 2)$  and  $D_+(2) = G_+(2; 0)$  are given in Example 4.8 below.

**Example 4.7** (*The rank-one case*). The *rank-one case* corresponds to triples  $(\mathfrak{a}, \Sigma, m)$  in which  $\mathfrak{a}$  is one-dimensional. Suppose  $\Sigma = \{\pm\alpha\}$  is of type  $A_1$ . Notice that  $P = \mathbb{Z}\alpha$  in this case. We identify  $\mathfrak{a}_{\mathbb{C}}$  and  $\mathfrak{a}_{\mathbb{C}}^*$  with  $\mathbb{C}$  by setting  $\lambda\alpha \equiv \lambda$  and  $zH_{\alpha}/2 \equiv z$  for  $\lambda, z \in \mathbb{C}$ . A multiplicity function can be identified with a complex number  $m \in \mathbb{C}$ . It is even if and only if  $m \in 2\mathbb{N}_0$ . We have then  $\rho(m) = m/2$ . The exponential function maps  $\mathfrak{a}_{\mathbb{C}} \equiv \mathbb{C}$  onto  $A_{\mathbb{C}} \equiv \mathbb{C}^{\times}$ , and  $A_{\mathbb{C}}^{\text{reg}} = \mathbb{C} \setminus \{0, \pm 1\}$ . The Weyl chamber  $\mathfrak{a}^+$  coincides with the half-line  $(0, \infty)$ . The Weyl group  $W$  reduces to  $\{1, -1\}$  and acts on  $\mathfrak{a}_{\mathbb{C}}$  by multiplication. We normalize the inner product so that  $\langle \alpha, \alpha \rangle = 1$ . With the identification of  $\mathbb{C}[A_{\mathbb{C}}] = \mathbb{C}[u, u^{-1}]$ , the Weyl denominator is given by  $\Delta(u) = u - u^{-1}$ . Set  $\theta := u \frac{d}{du}$ . Fix  $u^{-1} du$  as Haar measure on  $A \equiv \mathbb{R} \setminus \{0\}$ . Then (cf. [11], §3.3)

$$ML(m) = \theta^2 + m \frac{1 - u^{-2}}{1 + u^{-2}} \theta + \rho(m)^2,$$

$$G_-(-2; m) = \Delta(u)\theta + (m - 1)(u + u^{-1}),$$

$$G_+(2; m) = -\frac{1}{\Delta(u)}\theta \quad (\text{independent of } m).$$

Seen as differential operators on  $\mathfrak{a}_{\mathbb{C}} \equiv \mathbb{C}$ , we have with  $\Delta(z) := e^z - e^{-z}$ ,



$$\begin{aligned}
ML(m) &= \frac{d^2}{dz^2} + m \frac{1 - e^{-2z}}{1 + e^{-2z}} \frac{d}{dz} + \rho(m)^2, \\
G_-(-2; m) &= \Delta(z) \frac{d}{dz} + (m-1)(e^z + e^{-z}), \\
G_+(2; m) &= -\frac{1}{\Delta(z)} \frac{d}{dz}.
\end{aligned}$$

Moreover  $G_{\pm}(0; m) = \text{id}$ . For an arbitrary  $l \in 2\mathbb{N}$ , the shift operators  $G_{\pm}(\pm l)$  can be obtained by iteration according to Remark 4.6. In particular,

$$D_+(m) = G_+(m; 0) = G_+(2; 0)^{m/2}.$$

**Example 4.8** (*The complex case*). The complex case is characterized by the condition  $m_{\alpha} = 2$  for all  $\alpha \in \Sigma$ . As differential operators on  $A_{\mathbb{C}}$  or  $\mathfrak{a}_{\mathbb{C}}$  we have:

$$\begin{aligned}
D_-(2) &= G_-(-2; 2) = \sigma \left( \prod_{\alpha \in \Sigma^+} \partial(A_{\alpha}) \right) \circ \Delta, \\
D_+(2) &= G_+(2; 0) = \tilde{\sigma} \Delta^{-1} \prod_{\alpha \in \Sigma^+} \partial(A_{\alpha}),
\end{aligned}$$

where  $\sigma^{-1} := \prod_{\alpha \in \Sigma^+} \langle \alpha, \alpha \rangle$ ,  $\tilde{\sigma} := (-1)^{|\Sigma^+|} \sigma$ , and, as before,  $\Delta = \Delta(1)$ .

As proven by Heckman and Opdam, arbitrary shift operators can be computed using Cherednik operators.

**Proposition 4.9** ([9], Definition 4.2 and Proposition 4.4; [29], Theorem 5.13). *Let  $T(m; p)$  be the Cherednik operator associated with  $p \in S(\mathfrak{a}_{\mathbb{C}})$  and write*

$$T(m; p) = \sum_{w \in W} D_w(m; p) \otimes w$$

with  $D_w(m; p) \in \mathbb{D}(A_{\mathbb{C}}^{\text{reg}})$ . Let  $q(m) \in S(\mathfrak{a}_{\mathbb{C}})$  be defined by:

$$q(m; \lambda) := \prod_{\alpha \in \Sigma^+} \left( \lambda_{\alpha} + \frac{m_{\alpha}}{2} \right).$$

Then the fundamental shift operators  $G_+(2; m), G_-(-2; m) \in \mathbb{D}(A_{\mathbb{C}}^{\text{reg}})$  are given by:

$$\begin{aligned}
G_+(2; m) &= \Delta^{-1} \sum_{w \in W} D_w(m; q(m)), \\
G_-(-2; m+2) &= |W|^{-1} \sum_{w, v \in W} \varepsilon(w) v(D_w(m; q(m)) \Delta),
\end{aligned}$$

where  $\varepsilon : W \rightarrow \{\pm 1\}$  is the sign character. In particular, for  $f \in \mathbb{C}[A_{\mathbb{C}}]^W$  we have:

$$\begin{aligned} G_+(2; m)f &= \Delta^{-1}T(m; q(m))f, \\ G_-(-2; m+2)f &= |W|^{-1} \sum_{v \in W} v(T(m; q(m))\Delta)f. \end{aligned}$$

The remaining of this section is devoted to the proof of the following theorem, which is our first main result. It shows that multiplication of the shift operators  $D_{\pm}(m)$  by  $\Delta(m)$  yields differential operators with holomorphic coefficients on the entire  $A_{\mathbb{C}}$ .

**Theorem 4.10.** *Set:*

$$D_m := \Delta(m)D_+(m) \quad (= G_-^*(-m; m) = D_-(m)^*). \quad (32)$$

Then  $D_m$  is a  $W$ -invariant element of  $\mathbb{C}[A_{\mathbb{C}}] \otimes S(\mathfrak{a}_{\mathbb{C}})$ . All the coefficients of  $D_m$  vanish on  $\bigcup_{\alpha \in \Sigma^+} \{a \in A : \alpha(\log a) = 0\}$ .

For the proof of Theorem 4.10 we need to introduce some notation. Let  $\{H_1, \dots, H_l\}$  be a fixed orthonormal basis in  $\mathfrak{a}$  with dual basis  $\{\xi_1, \dots, \xi_l\}$  for  $\mathfrak{a}^*$ . The coordinates of  $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$  with respect to  $\{\xi_1, \dots, \xi_l\}$  are  $(\lambda_1, \dots, \lambda_l)$  with  $\lambda_j := \lambda(H_j)$ . For a multi-index  $I = (\iota_1, \dots, \iota_l) \in \mathbb{N}_0^l$  we adopt the common notation:

$$|I| = \sum_{j=1}^l \iota_j \quad \text{and} \quad I! = \prod_{j=1}^l \iota_j!$$

and set:

$$\lambda^I := \lambda_1^{\iota_1} \cdots \lambda_l^{\iota_l}.$$

If  $p := H_1^{\iota_1} \otimes \cdots \otimes H_l^{\iota_l} \in S(\mathfrak{a}_{\mathbb{C}})$ , then  $p(\lambda) = \lambda^I$ . The constant coefficient differential operator  $\partial(p)$  on  $A_{\mathbb{C}}$  (or  $\mathfrak{a}_{\mathbb{C}}$ ) will be denoted  $\partial_a^I$ , that is

$$\partial_a^I := \partial(p) = \partial(H_1)^{\iota_1} \cdots \partial(H_l)^{\iota_l}.$$

We set:

$$\partial_{\lambda}^I := \partial(\xi_1)^{\iota_1} \cdots \partial(\xi_l)^{\iota_l} = \left( \frac{\partial}{\partial \lambda_1} \right)^{\iota_1} \cdots \left( \frac{\partial}{\partial \lambda_l} \right)^{\iota_l}$$

for the constant coefficient differential operator on  $\mathfrak{a}_{\mathbb{C}}^*$  corresponding to  $\xi_1^{\iota_1} \otimes \cdots \otimes \xi_l^{\iota_l}$ . Observe that, for multi-indices  $I = (\iota_1, \dots, \iota_l)$  and  $K = (\kappa_1, \dots, \kappa_l)$ , one has:

$$\partial_a^I e^{\lambda} = \lambda^I e^{\lambda} \quad (33)$$

and

$$(\partial_\lambda^K \lambda^I)|_{\lambda=0} = K! \delta_{I,K} \quad (34)$$

with  $\delta_{I,K} = 1$  if  $\iota_j = \kappa_j$  for all  $j = 1, \dots, l$  and equal to 0 otherwise.

Since  $\Delta \in \mathbb{C}[A_{\mathbb{C}}]$ , every  $D \in \mathbb{C}_{\Delta}[A_{\mathbb{C}}] \otimes S(\mathfrak{a}_{\mathbb{C}})$  can be written as

$$D = \Delta^k \sum_I \omega_I \otimes \partial_a^I,$$

where  $\sum_I$  denotes a sum over a finite set of multi-indices  $I$ ,  $\omega_I \in \mathbb{C}[A_{\mathbb{C}}]$  and  $k \in \mathbb{Z}$ . We can assume that this representation of  $D$  has been chosen with  $k$  as big as possible. Then  $D \in \mathbb{C}[A_{\mathbb{C}}] \otimes S(\mathfrak{a}_{\mathbb{C}})$  if and only if  $k \in \mathbb{N}_0$ . (In particular, the condition  $k \in \mathbb{N}_0$  is independent of the choice of the basis  $\{H_1, \dots, H_j\}$  of  $\mathfrak{a}$  fixed for the representation of  $D$ .)

**Lemma 4.11.** *Suppose  $\Sigma$  is a reduced root system and  $\alpha, \beta \in \Sigma^+$  with  $\alpha \neq \beta$ . Then  $1 - e^{-2\alpha}$  and  $1 - e^{-2\beta}$  are relatively prime in  $\mathbb{C}[A_{\mathbb{C}}]$ .*

**Proof.** This follows from [3], Chapter VI, §3, Lemma 1(ii).  $\square$

**Lemma 4.12.** *Let  $D \in \mathbb{C}_{\Delta}[A_{\mathbb{C}}] \otimes S(\mathfrak{a}_{\mathbb{C}})$ . Suppose  $D = \Delta^k \sum_I \omega_I \otimes \partial_a^I$  (with  $\omega_I \in \mathbb{C}[A_{\mathbb{C}}]$  and  $k \in \mathbb{Z}$  maximal) is the representation of  $D$  with respect to the fixed basis  $\{H_1, \dots, H_l\}$  of  $\mathfrak{a}$ .*

- (a) *If all coefficients  $\Delta^k \omega_I$  are nonsingular on  $A$ , then  $D \in \mathbb{C}[A_{\mathbb{C}}] \otimes S(\mathfrak{a}_{\mathbb{C}})$ .*
- (b) *If  $D$  is  $W$ -invariant and all coefficients  $\Delta^k \omega_I$  are nonsingular on  $\overline{A^+}$ , then  $D \in \mathbb{C}[A_{\mathbb{C}}] \otimes S(\mathfrak{a}_{\mathbb{C}})$ .*
- (c) *If  $D$  is  $W$ -invariant and all coefficients  $\Delta^k \omega_I$  vanish on  $\partial(A^+)$ , then they also vanish on  $\bigcup_{\alpha \in \Sigma^+} \{a \in A : \alpha(\log a) = 0\}$ .*

**Proof.** By maximality of  $k$ , there is a multi-index  $I$  so that  $\Delta$  does not divide  $\omega_I$ . Hence there is  $\alpha \in \Sigma^+$  so that  $1 - e^{-2\alpha}$  does not divide  $\omega_I$ . If

$$\Delta^k \omega_I = e^{k\rho} (1 - e^{-2\alpha})^k \prod_{\beta \in \Sigma^+ \setminus \{\alpha\}} (1 - e^{-2\beta})^k \omega_I$$

is nonsingular on  $A$ , then  $k \in \mathbb{N}_0$  by Lemma 4.11. This proves (a).

For (b), notice first that the possible singularities of  $\Delta^k \omega_I$  in  $A$  lie in the zero set of  $\Delta$ , i.e., along the hypersurfaces  $\mathcal{H}_\alpha := \{a \in A : \alpha(\log a) = 0\}$  with  $\alpha \in \Sigma^+$ . By assumption, no singularities of the coefficients  $\Delta^k \omega_I$  lie in  $\overline{A^+}$ , that is, no hypersurface  $\mathcal{H}_\beta$  is singular when  $\beta \in \Pi$  is a positive simple root. Suppose  $\mathcal{H}_\alpha$  is a singular hypersurface of a coefficient  $\Delta^k \omega_K$  of  $D$ . There exist  $w \in W$  and  $\beta \in \Pi$  such that  $w\alpha = \beta$ . Since  $D$  is  $W$ -invariant, we have:

$$D = wD = \sum_I w(\Delta^k \omega_I) \otimes w\partial_a^I. \quad (35)$$

From (33) we obtain for arbitrarily fixed  $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ :

$$(De^\lambda)e^{-\lambda} = \Delta^k \sum_I \omega_I (\partial_a^I e^\lambda) e^{-\lambda} = \sum_I \Delta^k \omega_I \lambda^I.$$

Observe that

$$(w\partial_a^I)(e^\lambda) := w(\partial_a^I(w^{-1}e^\lambda)) = w(\partial_a^I e^{w^{-1}\lambda}) = w((w^{-1}\lambda)^I e^{w^{-1}\lambda}) = (w^{-1}\lambda)^I e^\lambda.$$

Hence (35) yields:

$$(De^\lambda)e^{-\lambda} = \sum_I w(\Delta^k \omega_I)(w\lambda)^I.$$

Therefore, for all  $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ ,

$$\sum_I w(\Delta^k \omega_I)(w\lambda)^I = \sum_I \Delta^k \omega_I \lambda^I.$$

Replacing  $\lambda$  with  $w\lambda$ , we conclude:

$$\sum_I w(\Delta^k \omega_I) \lambda^I = \sum_I \Delta^k \omega_I (w\lambda)^I,$$

and thus, by (34),

$$K!w(\Delta^k \omega_K) = \sum_I \Delta^k \omega_I (\partial_\lambda^K (w\lambda)^I)|_{\lambda=0}. \quad (36)$$

Since  $\Delta^k \omega_K$  is singular along  $\mathcal{H}_\alpha$ , it follows that  $w(\Delta^k \omega_K)$  is singular along  $w\mathcal{H}_\alpha = \mathcal{H}_\beta$ . According to (36),  $w(\Delta^k \omega_K)$  is a linear combination of the coefficients  $\Delta^k \omega_I$ . So there must be a multi-index  $I$  for which  $\Delta^k \omega_I$  is singular along  $\mathcal{H}_\beta$ , in contradiction to our assumption. Thus no coefficient  $\Delta^k \omega_I$  can be singular in  $A$ , and the claim follows then from (a).

Part (c) follows immediately from (36) and from the fact that  $W$  acts transitively on the Weyl chambers.  $\square$

The following key lemma is a consequence of a more general result, which will be proven in Appendix A (see Corollary A.9).

**Lemma 4.13.** *For every  $m \in \mathcal{M}^+$  and every multi-index  $I$  the function*

$$\Delta(m; a) \partial_\lambda^I \Phi(m; \lambda, a)|_{\lambda=0}$$

*extends continuously on  $\overline{A^+}$  by setting it equal to zero on the boundary  $\partial(A^+)$  of  $\overline{A^+}$ .*

**Proof of Theorem 4.10.** The  $W$ -invariance of  $D_m$  is an immediate consequence of Lemma 2.1 and the fact that  $D_+(m) = G_+(m; 0)$  is  $W$ -invariant. Lemma 2.1 also guarantees that  $D_m \in \mathbb{C}_\Delta[A_\mathbb{C}] \otimes S(\mathfrak{a}_\mathbb{C})$ . Hence we can write:

$$D_m = \Delta^k \sum_I \omega_I \otimes \partial_a^I, \quad (37)$$

where  $I$  are multi-indices,  $\omega_I \in \mathbb{C}[A_\mathbb{C}]$  and  $k \in \mathbb{Z}$  is maximal. Because of (33),

$$(D_m e^\lambda) e^{-\lambda} = \Delta^k \sum_I \omega_I \lambda^I. \quad (38)$$

From (34), (38), (29) and from the product rule for differentiation we obtain:

$$\begin{aligned} \Delta(a)^k \omega_I(a) &= \frac{1}{I!} \partial_\lambda^I \left( (D_m e^{\lambda(\log a)}) e^{-\lambda(\log a)} \right) \Big|_{\lambda=0} \\ &= \frac{1}{I!} \partial_\lambda^I \left( \frac{e^{-\lambda(\log a)}}{c_H^+(m; -\lambda)} \Delta(m; a) \Phi(m; \lambda, a) \right) \Big|_{\lambda=0} \\ &= \sum_{J+K=I} \frac{1}{J!K!} \partial_\lambda^J \left( \frac{e^{-\lambda(\log a)}}{c_H^+(m; -\lambda)} \right) \Big|_{\lambda=0} (\Delta(m; a) \partial_\lambda^K \Phi(m; \lambda, a)) \Big|_{\lambda=0}. \end{aligned}$$

Lemma 4.13 implies then that each coefficient  $\Delta^k \omega_I$  in the representation (37) of  $D_m$  extends continuously to  $\overline{A^+}$  by setting it equal to 0 on  $\partial(A^+)$ . Since  $D_m$  is  $W$ -invariant, the theorem thus follows from Lemma 4.12(b) and (c).  $\square$

In the following corollaries we shall always assume that  $m \in \mathcal{M}^+$  is a fixed even multiplicity function.

**Corollary 4.14.**  $D_-(m) := G_-(-m; m) \in \mathbb{C}[A_\mathbb{C}] \otimes S(\mathfrak{a}_\mathbb{C})$ .

**Remark 4.15.** Corollary 4.14 is consistent with the idea that, since  $G_-(-m; m)$  “removes” the singularities of  $\Phi(m; \lambda, a)$  along  $\partial(A^+)$ , it should not have singular coefficients there.

The following corollary will be important to determine formulas for the  $\Theta$ -hypergeometric functions in which, as in the complex case, an alternating sum appears (see Corollary 5.2). Its proof, which depends on Theorem 4.10 and also on some elementary algebraic properties of  $\mathbb{C}[A_\mathbb{C}]$ , can be found in Appendix B.

**Corollary 4.16.**

$$\Delta(m) G_+(m-2; 2) \circ \Delta(-1) \in \mathbb{C}[A_\mathbb{C}] \otimes S(\mathfrak{a}_\mathbb{C}).$$

Consequently  $\Delta(m) G_+(m-2; 2)$  is a  $W$ -invariant element of  $\mathbb{C}[A_\mathbb{C}] \otimes S(\mathfrak{a}_\mathbb{C})$ .

**Proof.** See Appendix B.  $\square$

**Corollary 4.17.** *Let  $U$  denote the tubular neighborhood of  $A$  from Theorem 3.1, and let  $D_m = \Delta(m)D_+(m)$  be as in Theorem 4.10. Set:*

$$\pi(\lambda) := \prod_{\alpha \in \Sigma^+} \lambda_\alpha. \quad (39)$$

*Then the polynomial  $\pi(\lambda)$  divides  $D_m e^{\lambda(\log a)}$  for all  $a \in U$ .*

**Proof.** By (31) and Example 4.8, we have:

$$\begin{aligned} D_m e^{\lambda(\log a)} &= \Delta(m)D_+(m)e^{\lambda(\log a)} \\ &= \Delta(m)G_+(m-2; 2)D_+(2)e^{\lambda(\log a)} \\ &= \Delta(m)G_+(m-2; 2) \left[ (-1)^{|\Sigma^+|} \Delta^{-1} \prod_{\alpha \in \Sigma^+} \frac{\partial(A_\alpha)}{\langle \alpha, \alpha \rangle} e^{\lambda(\log a)} \right] \\ &= \pi(\lambda) (-1)^{|\Sigma^+|} (\Delta(m)G_+(m-2; 2) \circ \Delta(-1)) e^{\lambda(\log a)}, \end{aligned}$$

and  $\Delta(m)G_+(m-2; 2) \circ \Delta(-1) \in \mathbb{C}[A_{\mathbb{C}}] \otimes S(\mathfrak{a}_{\mathbb{C}})$  by Corollary 4.16.  $\square$

## 5. Explicit formulas for the $\Theta$ -hypergeometric functions

In this section we prove explicit formulas for the  $\Theta$ -hypergeometric functions corresponding to even multiplicity functions. They generalize the well-known formula by Harish-Chandra for the spherical functions on Riemannian symmetric spaces of the noncompact type  $G/K$  with  $G$  complex and the formula by Faraut, Hilgert and Ólafsson for the spherical functions on NCC symmetric spaces  $G/H$  with  $G$  complex.

We introduce the polynomial:

$$e_{\Theta}^+(m; \lambda) = (-1)^{\frac{1}{2} \sum_{\alpha \in \langle \Theta \rangle^+} m_\alpha} \prod_{\alpha \in \langle \Theta \rangle^+} \prod_{k=-m_\alpha/2+1}^{m_\alpha/2-1} (\lambda_\alpha - k)$$

(with empty products equal to 1). Recall the polynomial  $e_{\Theta}^-(m; \lambda)$  and  $\pi(\lambda)$  from (21) and (39), respectively. Notice that

$$e_{\Theta}^-(m; \lambda) e_{\Theta}^+(m; \lambda) = (-1)^{d(\Theta, m)} e_{\Pi}^+(m; \lambda). \quad (40)$$

Finally, let

$$c_{\Theta}^{+,c}(m; \lambda) := \frac{c_{\Pi}^+(m; \lambda)}{c_{\Theta}^+(m; \lambda)} = \prod_{\alpha \in \Sigma^+ \setminus \langle \Theta \rangle^+} \prod_{k=0}^{m_\alpha/2-1} \frac{1}{\lambda_\alpha + k} = (-1)^{d(\Theta, m)} c_{\Theta}^-(m; \lambda).$$

**Theorem 5.1.** (a) Let  $m \in \mathcal{M}^+$  be even, and let  $D_m = \Delta(m)D_+(m) \in \mathbb{C}[A_{\mathbb{C}}] \times S(\mathfrak{a}_{\mathbb{C}})$  be as in (32). Then the  $\Theta$ -hypergeometric function  $\varphi_{\Theta}(m; \lambda, a)$  is determined by the formula:

$$e_{\Theta}^{-}(m; \lambda) \Delta(m; a) \varphi_{\Theta}(m; \lambda, a) = \frac{1}{\pi(\lambda) e_{\Theta}^{+}(m; \lambda)} D_m \left( \sum_{w \in W_{\Theta}} e^{w\lambda(\log a)} \right). \quad (41)$$

The left-hand side of (41) is a holomorphic function of  $(\lambda, a) \in \mathfrak{a}_{\mathbb{C}}^* \times A_{\Theta}U$ . For fixed  $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$  the function  $D_m(\sum_{w \in W_{\Theta}} e^{w\lambda(\log a)})$  is single valued and holomorphic in  $AU_{\Theta}^{\lambda}$ , where  $U_{\Theta}^{\lambda} (\supset U)$  is a largest open neighborhood of  $e$  in  $T$  in which  $e^{w\lambda(\log a)}$  is single-valued for all  $w \in W_{\Theta}$ . As a function of  $(\lambda, a)$ , it is single valued and holomorphic in  $\mathfrak{a}_{\mathbb{C}}^* \times U_{\Theta}$  with  $U_{\Theta} := \bigcap_{\lambda \in \mathfrak{a}_{\mathbb{C}}^*} U_{\Theta}^{\lambda}$ . We can take  $U_{\Theta}^{\lambda} = T$  when  $\lambda \in P$ .

(b) For all  $(\lambda, a) \in \mathfrak{a}_{\mathbb{C}}^* \times A_{\Theta}U$ , we have the following equality (as meromorphic functions of  $\lambda$  with singularities located along the zero set of  $e_{\Theta}^{-}(m; \lambda)$ ):

$$\Delta(m; a) \varphi_{\Theta}(m; \lambda, a) = (-1)^{d(\Theta, m)} \left[ \prod_{\alpha \in \Sigma^+} \prod_{k=0}^{m_{\alpha}/2-1} (k^2 - \lambda_{\alpha}^2) \right]^{-1} D_m \left( \sum_{w \in W_{\Theta}} e^{w\lambda(\log a)} \right) \quad (42)$$

with  $d(\Theta, m)$  as in (17).

(c) If  $\Theta = \Pi$ , then  $\Delta(m; a) \varphi_{\Pi}(m; \lambda, a)$  extends as a holomorphic function on  $\mathfrak{a}_{\mathbb{C}}^* \times AU_{\Pi}$  by means of the formula:

$$\Delta(m; a) \varphi_{\Pi}(m; \lambda, a) = \left[ \prod_{\alpha \in \Sigma^+} \prod_{k=0}^{m_{\alpha}/2-1} (k^2 - \lambda_{\alpha}^2) \right]^{-1} D_m \left( \sum_{w \in W} e^{w\lambda(\log a)} \right). \quad (43)$$

If, moreover,  $\lambda \in P$ , then  $\Delta(m; a) \varphi_{\Pi}(m; \lambda, a)$  extends by (43) as  $W$ -invariant entire function on  $A_{\mathbb{C}}$ .

**Proof.** To prove (a), observe first that, since  $c_{\Theta}^{+,c}(m; \lambda)$  and  $c_{\Theta}^{+}(m; \lambda) c_{\Theta}^{+}(m; -\lambda)$  are  $W_{\Theta}$ -invariant, we have, for all  $w \in W_{\Theta}$ ,

$$\begin{aligned} & c_{\Theta}^{-}(m; \lambda) c_{\Theta}^{+}(m; w\lambda) c_{\Pi}^{+}(m; -w\lambda) \\ &= c_{\Theta}^{-}(m; \lambda) c_{\Theta}^{+}(m; w\lambda) c_{\Theta}^{+}(m; -w\lambda) c_{\Theta}^{+,c}(m; -w\lambda) \\ &= c_{\Theta}^{-}(m; \lambda) c_{\Theta}^{+}(m; \lambda) c_{\Theta}^{+}(m; -\lambda) c_{\Theta}^{+,c}(m; -\lambda) \\ &= \left[ \left( \prod_{\alpha \in \Sigma^+ \setminus \langle \Theta \rangle^+} \lambda_{\alpha} \right) e_{\Theta}^{-}(m; \lambda) \left( \prod_{\alpha \in \langle \Theta \rangle^+} \lambda_{\alpha} \right) e_{\Theta}^{+}(m; \lambda) \right]^{-1} \\ &= [\pi(\lambda) e_{\Theta}^{-}(m; \lambda) e_{\Theta}^{+}(m; \lambda)]^{-1}. \end{aligned} \quad (44)$$

Recall from Theorem 3.6 that the function  $e_{\Theta}^{-}(m; \lambda) \varphi_{\Theta}(m; \lambda, a)$  is entire as a function of  $(\lambda, a) \in \mathfrak{a}_{\mathbb{C}}^* \times U_{\Theta}$ , where  $U_{\Theta}$  denotes a  $W_{\Theta}$ -invariant tubular neighborhood of  $A_{\Theta}$  in  $A_{\mathbb{C}}$ .

The definition of  $\Theta$ -hypergeometric functions, (29) and (44) give for all  $a \in A^+U$  and  $\lambda \in \mathfrak{a}_{\mathbb{C}}^* \setminus P$ :

$$\begin{aligned} \varphi_{\Theta}(m; \lambda, a) &= c_{\Theta}^{-}(m; \lambda) \sum_{w \in W_{\Theta}} c_{\Theta}^{+}(m; w\lambda) \Phi(m; w\lambda, a) \\ &= c_{\Theta}^{-}(m; \lambda) \sum_{w \in W_{\Theta}} c_{\Theta}^{+}(m; w\lambda) c_{\Pi}^{+}(m; -w\lambda) D_{+}(m) e^{w\lambda(\log a)} \\ &= \frac{1}{\pi(\lambda) e_{\Theta}^{-}(m; \lambda) e_{\Theta}^{+}(m; \lambda)} \sum_{w \in W_{\Theta}} D_{+}(m) e^{w\lambda(\log a)}. \end{aligned}$$

Hence

$$e_{\Theta}^{-}(m; \lambda) \Delta(m; a) \varphi_{\Theta}(m; \lambda, a) = \frac{1}{\pi(\lambda) e_{\Theta}^{+}(m; \lambda)} D_m \left( \sum_{w \in W_{\Theta}} e^{w\lambda(\log a)} \right).$$

Since  $D_m \in \mathbb{C}[A_{\mathbb{C}}] \otimes S(\mathfrak{a}_{\mathbb{C}})$  is  $W$ -invariant, the formula extends by  $W_{\Theta}$ -invariance to  $\mathfrak{a}_{\mathbb{C}}^* \times A_{\Theta}U$ .

Parts (b) and (c) follow from (a) by easy computations.  $\square$

**Corollary 5.2.** *Let  $m \in \mathcal{M}^+$  be even. Then there exist differential operators  $G_m, \tilde{G}_m \in \mathbb{C}[A_{\mathbb{C}}] \otimes S(\mathfrak{a}_{\mathbb{C}})$  with  $\tilde{G}_m = G_m \circ \Delta$  and  $\tilde{G}_m$   $W$ -invariant such that for all  $(\lambda, a) \in \mathfrak{a}_{\mathbb{C}}^* \times A_{\Theta}U$  the following equality of meromorphic functions of  $\lambda$  holds:*

$$\begin{aligned} \Delta(m; a) \varphi_{\Theta}(m; \lambda, a) &= \frac{(-1)^{d(\Theta, m)}}{\pi(\lambda)} \left[ \prod_{\alpha \in \Sigma^+} \prod_{k=1}^{m_{\alpha}/2-1} (k^2 - \lambda_{\alpha}^2) \right]^{-1} \\ &\quad \times G_m \left( \sum_{w \in W_{\Theta}} \varepsilon(w) e^{w\lambda(\log a)} \right) \\ &= (-1)^{d(\Theta, m-2)} \left[ \prod_{\alpha \in \Sigma^+} \prod_{k=1}^{m_{\alpha}/2-1} (k^2 - \lambda_{\alpha}^2) \right]^{-1} \tilde{G}_m \varphi_{\Theta}(2; \lambda, a). \end{aligned}$$

If  $m = 2$ , then  $G_m = \Delta$ . If  $\Theta = \Pi$ , then

$$\begin{aligned} \Delta(m; a) \varphi_{\Pi}(m; \lambda, a) &= \left[ \pi(\lambda) \prod_{\alpha \in \Sigma^+} \prod_{k=1}^{m_{\alpha}/2-1} (k^2 - \lambda_{\alpha}^2) \right]^{-1} G_m \left( \sum_{w \in W} \varepsilon(w) e^{w\lambda(\log a)} \right) \\ &= \left[ \prod_{\alpha \in \Sigma^+} \prod_{k=1}^{m_{\alpha}/2-1} (k^2 - \lambda_{\alpha}^2) \right]^{-1} \tilde{G}_m \varphi_{\Pi}(2; \lambda, a), \end{aligned}$$



in which the right-hand sides extend as holomorphic functions of  $(a, \lambda) \in AU \times \mathfrak{a}_{\mathbb{C}}^*$ . If  $\lambda \in P$  is fixed, then the right-hand sides of the previous equalities extend as holomorphic functions of  $a \in A_{\mathbb{C}}$ .

**Proof.** Set  $G_m := \Delta(m)G_+(m-2; 2) \circ \Delta(-1)$ . Then the formulas follow immediately from Theorem 5.1, Corollary 4.16 and the equalities in the proof of Corollary 4.17 (see also Example 5.4 below).  $\square$

**Example 5.3** (*The rank-one case*). In the rank-one case the only two possibilities are  $\Theta = \emptyset$  and  $\Theta = \Pi$ . If  $\Theta = \emptyset$ , we have a multiple (with a function of  $\lambda$  as a constant) of the Harish-Chandra series:

$$\Phi(m; \lambda, z) = 2^{-m/2} \prod_{k=0}^{m/2-1} (\lambda - k) \left( \frac{1}{\sinh z} \frac{d}{dz} \right)^{m/2} e^{\lambda z}. \quad (45)$$

In particular,

$$\Phi(2; \lambda, z) = \frac{1}{\Delta(z)} e^{\lambda z}.$$

If  $\Theta = \Pi$ , then

$$\frac{\varphi_{\Pi}(m; \lambda, z)}{c_{\Pi}^+(m; \rho)} = 2^{m/2-1} \prod_{k=0}^{m/2-1} \frac{(m/2 + k)}{(\lambda^2 - k^2)} \left( \frac{1}{\sinh z} \frac{d}{dz} \right)^{m/2} \cosh(\lambda z) \quad (46)$$

is the formula for Harish-Chandra's spherical functions on the Riemannian symmetric spaces  $SO_0(1, n)/SO(n)$  (with  $n = m + 1$ ), as found by Takahashi (see [34], Formula (21), p. 326). In particular,

$$\frac{\varphi_{\Pi}(2; \lambda, z)}{c_{\Pi}^+(2; \rho)} = \frac{1}{\lambda} \frac{\sinh(\lambda z)}{\sinh z}$$

in the rank-one complex case.

**Example 5.4** (*The complex case*). In this case  $d(\Theta, m) = |\Sigma^+ \setminus \langle \Theta \rangle^+|$ . Since

$$\prod_{\alpha \in \Sigma^+} \partial(A_{\alpha}) e^{w\lambda(\log a)} = \left( \prod_{\alpha \in \Sigma^+} \langle w\lambda, \alpha \rangle \right) e^{w\lambda(\log a)} = \varepsilon(w) \left( \prod_{\alpha \in \Sigma^+} \langle \lambda, \alpha \rangle \right) e^{w\lambda(\log a)},$$

we obtain:

$$\varphi_{\Theta}(2; \lambda, a) = (-1)^{|\Sigma^+ \setminus \langle \Theta \rangle^+|} \frac{1}{\pi(\lambda)} \frac{1}{\Delta(a)} \sum_{w \in W_{\Theta}} \varepsilon(w) e^{w\lambda(\log a)}. \quad (47)$$

In particular,

$$\frac{\varphi_{\Pi}(2; \lambda, a)}{c_{\Pi}^{+}(2; \rho)} = \frac{\pi(\rho)}{\pi(\lambda)} \frac{1}{\Delta(a)} \sum_{w \in W} \varepsilon(w) e^{w\lambda(\log a)} \quad (48)$$

gives Harish-Chandra's formula for the spherical functions on Riemannian symmetric spaces  $G/K$  with  $G$  complex, and (with  $W_0 := W_{\Pi_0}$ )

$$\frac{\varphi_{\Pi_0}(2; \lambda, a)}{c_{\Pi_0}^{+}(2; \rho) c_{\Pi_0}^{-}(2; \rho)} = \frac{\pi(\rho)}{\pi(\lambda)} \frac{1}{\Delta(a)} \sum_{w \in W_0} \varepsilon(w) e^{w\lambda(\log a)} \quad (49)$$

gives the formula for the spherical functions on NCC symmetric spaces  $G/H$ , where  $G$  is complex, as in [6].

## 6. Some remarks on the Paley–Wiener space

Before proceeding with the proof of Theorem 1.3, let us add some remarks on the definition of the Paley–Wiener space. Let  $m \in \mathcal{M}^{+}$  be an even multiplicity function,  $\Theta \subset \Pi$  a set of positive simple roots, and  $C$  a compact, convex and  $W_{\Theta}$ -invariant subset of  $\mathfrak{a}_{\Theta}$ . Recall from Definition 1.2 that the Paley–Wiener space  $\text{PW}_{\Theta}(m; C)$  is the space of all  $W_{\Theta}$ -invariant meromorphic functions  $g: \mathfrak{a}_{\mathbb{C}}^{*} \rightarrow \mathbb{C}$  satisfying the following properties:

- (1) The function  $e_{\Theta}^{-}(m; \lambda)g(\lambda)$  is entire of exponential type  $C$  and rapidly decreasing, that is for every  $N \in \mathbb{N}$  there is a constant  $C_N \geq 0$  such that

$$|e_{\Theta}^{-}(m; \lambda)g(\lambda)| \leq C_N (1 + |\lambda|)^{-N} e^{q_C(\text{Re } \lambda)}$$

for all  $\lambda \in \mathfrak{a}_{\mathbb{C}}^{*}$ .

- (2) The function

$$\text{P}_{\Theta}^{\text{av}} g(\lambda) := \sum_{w \in W_{\Theta} \setminus W} g(w\lambda)$$

extends to an entire function on  $\mathfrak{a}_{\mathbb{C}}^{*}$ .

**Proposition 6.1.** *Condition (2) above is automatically satisfied in the following cases:*

- (a) *Euclidean case* ( $m = 0$ ,  $\Theta$  arbitrary);
- (b) *Heckman–Opdam case* ( $\Theta = \Pi$ ,  $m$  arbitrary);
- (c) *Complex case* ( $m = 2$ ,  $\Theta$  arbitrary).

**Proof.** In the first two cases we have  $e_{\Theta}^{-}(m; \lambda) = 1$ , hence all functions in  $\text{PW}_{\Theta}(m; C)$  are entire. In the complex case  $e_{\Theta}^{-}(2; \lambda) = \prod_{\alpha \in \Sigma^{+} \setminus \langle \Theta \rangle} \lambda_{\alpha}$ . Condition 1 in the definition of the

Paley–Wiener space implies that every  $g \in \text{PW}_\Theta(m; C)$  has at most a first order pole along each hyperplane  $\lambda_\alpha = 0$ . The same is then true for  $\text{P}_\Theta^{\text{av}} g$ . But  $\text{P}_\Theta^{\text{av}} g$  is  $W$ -invariant, so it cannot have first order poles on hyperplanes  $\lambda_\alpha = 0$  with  $\alpha \in \Sigma^+$ . They must be therefore removable singularities, i.e., Condition (2) holds.  $\square$

**Remark 6.2.** Condition (2) in Definition 1.2 does not follow in general from Condition (1). For instance, in the rank-one case with  $m = 4$  and the usual identification of  $\mathfrak{a}$  with  $\mathbb{R}$ , we have  $e_\Theta^-(4; \lambda) = \lambda(\lambda - 1)(\lambda + 1)$ . If  $h \in \text{PW}(C)$ , where  $C$  is an arbitrary compact interval in  $(0, +\infty)$ , then the meromorphic function

$$g(\lambda) = \left( \frac{a}{\lambda - 1} + \frac{b}{\lambda + 1} \right) h(\lambda)$$

satisfies the first condition, but not the second when  $a \neq -b$ .

Let  $\ell$  denote the length function on  $W$  with respect to the simple reflections  $r_\beta$  with  $\beta \in \Pi$ . For  $\Theta \subset \Pi$  set:

$$W^\Theta := \{w \in W : \ell(wr_\alpha) > \ell(w) \text{ for all } \alpha \in \Theta\}. \quad (50)$$

Then every element  $w \in W$  can be uniquely written as  $w = uv$  with  $u \in W^\Theta$  and  $v \in W_\Theta$  (see, e.g., [17], p. 19). The element  $u$  is characterized as the unique element of smallest length in the coset  $wW_\Theta$ . Hence a set of coset representatives for  $W_\Theta \backslash W$  consists of the elements of  $\{u^{-1} : u \in W^\Theta\}$ . In particular we obtain:

$$\text{P}_\Theta^{\text{av}} g(\lambda) = \sum_{u \in W^\Theta} g(u^{-1}\lambda). \quad (51)$$

**Lemma 6.3.** *Let be  $W^\Theta$  as in (50). Then for  $u, u' \in W^\Theta$  with  $u \neq u'$  one has*

$$u\mathfrak{a}_\Theta \cap u'\mathfrak{a}_\Theta = \emptyset.$$

**Proof.** Since  $W$  acts simply transitively on the Weyl chambers, there are  $|W|$  Weyl chambers. A nonempty intersection of  $u\mathfrak{a}_\Theta$  and  $u'\mathfrak{a}_\Theta$  would contain at least a Weyl chamber. This is not possible since  $|W| = |W^\Theta||W_\Theta|$ .  $\square$

The following lemma, which is an easy modification of Lemma 5.13 in [14], will be applied several times in the sequel.

**Lemma 6.4.** *Let  $C$  be a compact convex subset of  $\mathfrak{a} \equiv \mathbb{R}^l$ , and let  $g : \mathfrak{a}_\mathbb{C}^* \equiv \mathbb{C}^l \rightarrow \mathbb{C}$  belong to the Paley–Wiener space  $\text{PW}(C)$ . Suppose  $q$  is a polynomial so that  $h := g/q$  is entire. Then  $h \in \text{PW}(C)$ .*

In the following we denote by  $\text{conv}(W(C))$  the closed convex hull of the  $W$ -orbit of a subset  $C$  of  $\mathfrak{a}$ . Moreover, we normalize the Lebesgue measure on  $i\mathfrak{a}^*$  so that the inverse transform to  $\mathcal{F}_A$  is given by:

$$\mathcal{F}_A^{-1}g(a) := \int_{i\mathfrak{a}^*} g(\lambda) e^{-\lambda(\log a)} d\lambda.$$

**Proposition 6.5.** *Let  $C \subset \mathfrak{a}_\Theta$  be compact, convex and  $W_\Theta$ -invariant. Suppose  $g \in \text{PW}_\Theta(m; C)$ . Then  $P_\Theta^{\text{av}}g \in \text{PW}(\text{conv}(W(C)))^W$ . Moreover, the map*

$$P_\Theta^{\text{av}} : \text{PW}_\Theta(m; C) \rightarrow \text{PW}(\text{conv}(W(C)))^W$$

*is linear and injective.*

**Proof.** The first statement is an immediate consequence of Lemma 6.4 with  $f := P_\Theta^{\text{av}}g$  and

$$q(\lambda) := \prod_{\alpha \in \Sigma} \prod_{k=-m_\alpha/2+1}^{m_\alpha/2-1} (\lambda_\alpha - k). \quad (52)$$

Indeed  $q$  is  $W$ -invariant and  $q(\lambda)P_\Theta^{\text{av}}g(\lambda) = \sum_{w \in W_\Theta \setminus W} q(\lambda)g(w\lambda)$ , with  $q(\lambda)g(w\lambda) \in \text{PW}(wC)$  for all  $w \in W$ .

To prove the injectivity of  $P_\Theta^{\text{av}}$ , suppose  $g \in \text{PW}_\Theta(m; C)$  satisfies  $P_\Theta^{\text{av}}g = 0$ . With  $q$  as above, one has  $qg \in \text{PW}(C)^{W_\Theta}$ , and  $P_\Theta^{\text{av}}(qg) = qP_\Theta^{\text{av}}g = 0$ . Hence for all  $a \in A$ ,

$$\begin{aligned} 0 &= \mathcal{F}_A^{-1}(P_\Theta^{\text{av}}(qg))(a) \\ &= \sum_{w \in W_\Theta \setminus W} \int_{i\mathfrak{a}^*} q(w\lambda)g(w\lambda) e^{-\lambda(\log a)} d\lambda \\ &= \sum_{w \in W_\Theta \setminus W} \int_{i\mathfrak{a}^*} q(\lambda)g(\lambda) e^{-\lambda(\log wa)} d\lambda \\ &= \sum_{w \in W_\Theta \setminus W} \mathcal{F}_A^{-1}(qg)(wa) \\ &= \sum_{u \in W^\Theta} \mathcal{F}_A^{-1}(qg)(u^{-1}a). \end{aligned} \quad (53)$$

By the classical Paley–Wiener theorem:

$$\text{supp}[\mathcal{F}_A^{-1}(qg) \circ u^{-1}] \subset u \exp C \subset uA_\Theta.$$

Lemma 6.3 and (53) therefore imply  $\mathcal{F}_A^{-1}(qg) \circ u^{-1} = 0$  for all  $u \in W^\Theta$ . In particular,  $\mathcal{F}_A^{-1}(qg) = 0$ . Thus  $qg = 0$ , which proves  $g = 0$  because  $g$  is meromorphic.  $\square$

**Corollary 6.6.** *Let  $q$  be as in (52), and let  $\iota_\Theta : A_\Theta \hookrightarrow A$  be the inclusion map. Then*

$$\left( \frac{1}{q} \mathcal{F}_A \circ \iota_\Theta \circ \mathcal{F}_A^{-1} \circ q \right) \circ \mathbf{P}_\Theta^{\text{av}} = \text{id}_{\text{PW}_\Theta(m; C)}.$$

**Proof.** As in the proof of Proposition 6.5, we have for  $g \in \text{PW}_\Theta(m; C)$ ,

$$\begin{aligned} \iota_\Theta \mathcal{F}_A^{-1}(q \mathbf{P}_\Theta^{\text{av}} g) &= \iota_\Theta \mathcal{F}_A^{-1} \mathbf{P}_\Theta^{\text{av}}(qg) \\ &= \iota_\Theta \left( \sum_{u \in W^\Theta} (\mathcal{F}_A^{-1}(qg) \circ u^{-1}) \right) \\ &= \mathcal{F}_A^{-1}(qg). \quad \square \end{aligned}$$

## 7. The $\Theta$ -hypergeometric transform

Recall from (2) that the  $\Theta$ -hypergeometric transform of a sufficiently regular  $W_\Theta$ -invariant function  $f : A_\Theta \rightarrow \mathbb{C}$  is the  $W_\Theta$ -invariant function  $\mathcal{F}_\Theta f(m)$  on  $\mathfrak{a}_\mathbb{C}^*$  defined by:

$$\mathcal{F}_\Theta f(m; \lambda) := \frac{1}{|W_\Theta|} \int_{A_\Theta} f(a) \varphi_\Theta(m; \lambda, a) \Delta(m; a) da$$

with  $\Delta(m)$  as in (3).

**Lemma 7.1.** *If  $f \in C_c^\infty(A_\Theta)^{W_\Theta}$ , then  $e_\Theta^-(m; \lambda) \mathcal{F}_\Theta f(m; \lambda)$  is a  $W_\Theta$ -invariant entire function on  $\mathfrak{a}_\mathbb{C}^*$ .*

**Proof.** This follows immediately from Theorem 3.6.  $\square$

**Corollary 7.2.** *Every function  $f \in C_c^\infty(A_\Theta)^{W_\Theta}$  can be uniquely extended to a  $W$ -invariant function  $f_\Pi \in C_c^\infty(A)^W$ . Moreover,*

$$(\mathcal{F}_\Pi f_\Pi)(m; \lambda) = (-1)^{d(\Theta, m)} \sum_{W_\Theta \setminus W} (\mathcal{F}_\Theta f)(m; w\lambda) = (-1)^{d(\Theta, m)} (\mathbf{P}_\Theta^{\text{av}} \mathcal{F}_\Theta f)(m; \lambda),$$

where  $d(\Theta, m)$  is as in (17).

**Proof.** Immediate consequence of Lemmas 6.3 and 3.8.  $\square$

The inversion formula for the NCC spaces was first proven in [21]. The general case was treated in [33] (see also [31]). Observing that in the even multiplicity case,

$$\pi(\lambda) e_\Pi^+(m; \lambda) = \frac{1}{|c_\Pi^+(m; \lambda)|^2}, \quad \lambda \in \mathfrak{ia}^*,$$

we can state it as follows.

**Theorem 7.3** ([33], Theorem 4.5; see also [31], Theorem 2.4.5). *Let  $m \in \mathcal{M}^+$  be an even multiplicity function, and let  $f \in C_c(A_\Theta)^{W_\Theta}$ . Then there is a constant  $k > 0$  (depending only on the normalization of the measures) so that for all  $f \in C_c(A_\Theta)^{W_\Theta}$  the following inversion formula holds: For all  $a \in A_\Theta$ ,*

$$\begin{aligned} f(a) &= (-1)^{d(\Theta, m)} k \frac{|W|}{|W_\Theta|} \int_{\mathfrak{ia}^*} (\mathcal{F}_\Theta f)(m; \lambda) \varphi_\Pi(m; -\lambda, a) \frac{d\lambda}{|c_\Pi^+(m; \lambda)|^2} \\ &= (-1)^{d(\Theta, m)} k \frac{|W|}{|W_\Theta|} \int_{\mathfrak{ia}^*} (\mathcal{F}_\Theta f)(m; \lambda) \pi(\lambda) e_\Pi^+(m; \lambda) \varphi_\Pi(m; -\lambda, a) d\lambda. \end{aligned}$$

## 8. Transforms of compactly supported smooth functions

In this section we begin the proof of the Paley–Wiener theorem (Theorem 1.3) by showing that the  $\Theta$ -hypergeometric transform maps  $C_c^\infty(C)^{W_\Theta}$  into the Paley–Wiener space  $\text{PW}_\Theta(m; C)$ . The key property is the fact that  $D_m = \Delta(m)D_+(m) = D_-(m)^*$  is a  $W$ -invariant differential operator on  $A$  with smooth coefficients.

**Theorem 8.1.** *Let  $C \subset A_\Theta$  be compact, convex and  $W_\Theta$ -invariant. Then the  $\Theta$ -hypergeometric transform maps  $C_c^\infty(C)^{W_\Theta}$  injectively into  $\text{PW}_\Theta(m; C)$ .*

**Proof.** Suppose  $f \in C_c^\infty(C)^{W_\Theta}$ . Because of Theorem 4.10 we have,

$$\begin{aligned} &\pi(\lambda) e_\Theta^+(m; \lambda) e_\Theta^-(m; \lambda) \mathcal{F}_\Theta f(m; \lambda) \\ &= \frac{1}{|W_\Theta|} \int_{A_\Theta} f(a) \pi(\lambda) e_\Pi^+(m; \lambda) \varphi_\Theta(m; \lambda, a) \Delta(m; a) da \\ &= \frac{1}{|W_\Theta|} \int_{A_\Theta} f(a) \sum_{w \in W_\Theta} D_m e^{w\lambda(\log a)} da \\ &= \frac{1}{|W_\Theta|} \sum_{w \in W_\Theta} \int_{A_\Theta} f(a) D_-(m)^* e^{w\lambda(\log a)} da \\ &= \frac{1}{|W_\Theta|} \sum_{w \in W_\Theta} \int_{A_\Theta} [D_-(m) f(a)] e^{\lambda(\log w^{-1}a)} da \\ &= \int_{A_\Theta} [D_-(m) f(a)] e^{\lambda(\log a)} da \\ &= [\mathcal{F}_A(D_-(m) f)](\lambda). \end{aligned}$$

In the above equalities we have used the fact that  $D_-(m)$  is a  $W$ -invariant differential operator with smooth coefficients on  $A$ , which implies that  $D_-(m)f(a)$  is a  $W_\Theta$ -invariant smooth function on  $A$  with compact support in  $\exp C \subset A_\Theta$ . The classical Paley–Wiener theorem for the Fourier transform yields then that

$$\pi_\Pi(m; \lambda) e_\Theta^+(m; \lambda) e_\Theta^-(m; \lambda) \mathcal{F}_\Theta f(m; \lambda)$$

belongs to  $\text{PW}(C)$ . By Lemma 7.1 the function  $e_\Theta^-(m; \lambda) \mathcal{F}_\Theta f(m; \lambda)$  is entire and  $W_\Theta$ -invariant. It is therefore an element of  $\text{PW}(C)^{W_\Theta}$  by Lemma 6.4. Furthermore, Corollary 7.2 gives  $\text{P}_\Theta^{\text{av}} \mathcal{F}_\Theta f = (-1)^{d(\Theta, m)} \mathcal{F}_\Pi f_\Pi$ , which is entire again by Lemma 7.1. This proves that  $\mathcal{F}_\Theta f(m) \in \text{PW}_\Theta(m; C)$ .

The injectivity of  $\mathcal{F}_\Theta$  on  $C_c^\infty(C)^{W_\Theta}$  follows from the inversion formula in Theorem 7.3.  $\square$

## 9. Wave packets

**Definition 9.1.** Let  $m \in \mathcal{M}^+$  be a fixed even multiplicity function. The *wave-packet* of  $g : \mathfrak{a}_\mathbb{C}^* \rightarrow \mathbb{C}$  is the function  $\mathcal{I}g = \mathcal{I}g(m) : A \rightarrow \mathbb{C}$  defined by:

$$\begin{aligned} (\mathcal{I}g)(a) &:= \int_{\mathfrak{ia}^*} g(\lambda) \varphi_\Pi(m; -\lambda, a) \frac{d\lambda}{|c_\Pi^+(m; \lambda)|^2} \\ &= \int_{\mathfrak{ia}^*} g(\lambda) \pi(\lambda) e_\Pi^+(m; \lambda) \varphi_\Pi(m; -\lambda, a) d\lambda, \end{aligned} \quad (54)$$

provided the integrals converge. In this case, the  $\Theta$ -wave-packet of  $g$  is the function on  $A_\Theta$  obtained by restriction of  $\mathcal{I}g$  to  $A_\Theta$ , that is

$$\mathcal{I}_\Theta g = \mathcal{I}g \circ \iota_\Theta, \quad (55)$$

where  $\iota_\Theta : A_\Theta \hookrightarrow A$  is the inclusion map. Hence  $\mathcal{I} = \mathcal{I}_\Pi$ .

**Remark 9.2.** Suppose the integrals (54) converge for  $g : A \rightarrow \mathbb{C}$ . The  $W$ -invariance of  $\varphi_\Pi(m; -\lambda, a)$  in  $a \in A$  implies that  $\mathcal{I}g$  is  $W$ -invariant. Furthermore, by  $W$ -invariance in the  $\lambda$ -variable of  $\varphi_\Pi(m; -\lambda, a)$  and  $|c_\Pi^+(m; \lambda)|^2$ , we have  $\mathcal{I}g = \frac{|W_\Theta|}{|W|} \mathcal{I} \text{P}_\Theta^{\text{av}} g$ .

**Lemma 9.3.** Let  $m \in \mathcal{M}^+$  be a fixed even multiplicity function. Let  $C$  be a compact convex subset of  $\mathfrak{a}$ . Suppose that  $g : \mathfrak{a}_\mathbb{C}^* \rightarrow \mathbb{C}$  satisfies  $\pi(\lambda) e_\Pi^+(m; \lambda) g(\lambda) \in \text{PW}(C)$ . Then  $\mathcal{I}g$  is a well-defined  $W$ -invariant smooth function on  $A$ .

In particular, if  $C$  is  $W_\Theta$ -invariant and  $g \in \text{PW}_\Theta(m; C)$ , then  $\mathcal{I}_\Theta g$  is a well-defined  $W_\Theta$ -invariant smooth function on  $A_\Theta$ .

**Proof.** The assumption guarantees that for all  $N \in \mathbb{N}$  there is a constant  $C_N > 0$  for which  $|\pi(\lambda)e_{\Pi}^+(m; \lambda)g(\lambda)| \leq C_N(1 + |\lambda|)^{-N}$  for all  $\lambda \in i\mathfrak{a}^*$ . Because of (23), we can therefore differentiate (54) under integral sign.  $\square$

**Lemma 9.4.** *Let  $C \subset \mathfrak{a}_{\Theta}$  be compact, convex and  $W_{\Theta}$ -invariant, and let  $g \in \text{PW}_{\Theta}(m; C)$ . Then  $\text{supp } \mathcal{I}g \subset \text{conv}(W(C))$ .*

**Proof.** By Remark 9.2 we have  $\mathcal{I}g = \frac{|W_{\Theta}|}{|W|} \mathcal{I}P_{\Theta}^{\text{av}} g$  with  $P_{\Theta}^{\text{av}} g \in \text{PW}(\text{conv}(W(C)))^W$ . Formulas (41) and (54) give for all  $h \in \text{PW}(\text{conv}(W(C)))^W$  and  $a \in A$ ,

$$\begin{aligned} \Delta(m; a)(\mathcal{I}h)(a) &= \int_{i\mathfrak{a}^*} h(\lambda) \pi(\lambda) e_{\Pi}^+(m; \lambda) \varphi_{\Pi}(m; -\lambda, a) d\lambda \\ &= \int_{i\mathfrak{a}^*} h(\lambda) \sum_{w \in W} D_m e^{w\lambda(\log a)} d\lambda \\ &= \sum_{w \in W} D_m \int_{i\mathfrak{a}^*} h(\lambda) e^{w\lambda(\log a)} d\lambda \\ &= |W| (D_m \mathcal{F}_A^{-1} h)(a). \end{aligned}$$

The classical Paley–Wiener theorem implies that  $\mathcal{F}_A^{-1}h$ , and hence  $\Delta(m)\mathcal{I}h$ , has support in  $\text{conv}(W(C))$ . Since  $\mathcal{I}h$  is smooth, this implies the claim.  $\square$

**Remark 9.5.** An alternative proof of Lemma 9.4, which indeed holds for the  $\Theta$ -hypergeometric transform corresponding to an arbitrary multiplicity function  $m \in \mathcal{M}^+$ , can be obtained via the equality  $\mathcal{I}g = \frac{|W_{\Theta}|}{|W|} \mathcal{I}P_{\Theta}^{\text{av}} g$  and the Paley–Wiener theorem for the Opdam transform (see Theorem 10.1 below). The proof given above requires however only elementary tools.

## 10. The case $\Theta = \Pi$

In the case  $\Theta = \Pi$  our spherical transform coincides (up to a constant multiple depending on the multiplicities and on the normalization of the measures) with the Opdam transform. For the latter transform, the following version of the Paley–Wiener theorem was proven for arbitrary  $m \in \mathcal{M}^+$  in [28].<sup>3</sup>

**Theorem 10.1** ([28], Theorems 8.6 and 9.13(4); see also [29], p. 49). *Let  $m \in \mathcal{M}^+$  be arbitrarily fixed. Let  $H \in \mathfrak{a}$  and  $C(H) := \text{conv}(W(H))$ . Then  $\mathcal{F}_{\Pi}$  maps  $C_c^{\infty}(C(H))^W$  bijectively onto  $\text{PW}_{\Pi}(m; C(H)) \equiv \text{PW}(C(H))^W$ . The inverse of  $\mathcal{F}_{\Pi}$  is  $k\mathcal{I}_{\Pi}$ , where  $k$  is the normalizing constant appearing in Theorem 7.3.*

<sup>3</sup> For arbitrary multiplicity functions  $m \in \mathcal{M}^+$ , the  $\Theta$ -hypergeometric transform is defined by (2) with  $\Delta(m) := \prod_{\alpha \in \Sigma^+} |e^{\alpha} - e^{-\alpha}|^{m_{\alpha}}$ .



**Corollary 10.2.** *Let  $m \in \mathcal{M}^+$  be arbitrary, and let  $C$  be a compact, convex and  $W$ -invariant subset of  $\mathfrak{a}$ . Then  $\mathcal{F}_\Pi$  maps  $C_c^\infty(C)^W$  bijectively onto  $\text{PW}_\Pi(m; C) \equiv \text{PW}(C)^W$ . The inverse transform is  $k\mathcal{I}_\Pi$ , where  $k$  is the normalizing constant appearing in Theorem 7.3.*

**Proof.** Since  $C \subset C(H)$  for suitable  $H \in \mathfrak{a}$  and  $\mathcal{F}_\Pi$  is bijective with inverse  $k\mathcal{I}_\Pi$  on  $C_c^\infty(C(H))^W$ , we only need to show that  $\mathcal{F}_\Pi$  maps  $C_c^\infty(C)^W$  onto  $\text{PW}(C)^W$ . One can argue as in Theorem 8.6 (1) of [28], that is using the estimates (23) together with the fact that  $\varphi_\Pi(m; \lambda)$  satisfies the hypergeometric system of differential equations (14). See also Proposition 8.3 in [2].

Conversely, suppose  $g \in \text{PW}(C)^W$ . The classical Paley–Wiener theorem ensures that  $\mathcal{F}_A^{-1}g \in C_c^\infty(C)^W$ . Let  $\varepsilon > 0$ , and let  $U_\varepsilon(C)$  denote the closed  $\varepsilon$ -neighborhood of  $C$ . Then we can find  $H_1, \dots, H_n \in \mathfrak{a}$  so that  $C \subset \bigcup_{j=1}^n C(H_j)^0 \subset U_\varepsilon(C)$ . Here  $C(H)^0$  denotes the interior of  $C(H)$ . Let  $\{\chi_1, \dots, \chi_n\}$  be a smooth partition of unity associated with the open covering  $\{C(H_1)^0, \dots, C(H_n)^0\}$  of  $C$ . Hence  $\chi_j \geq 0$  with  $\text{supp } \chi_j \subset C(H_j)^0$  for all  $j = 1, \dots, n$ , and  $\sum_{j=1}^n \chi_j = 1$  on  $C$ . Since  $\mathcal{F}_A^{-1}g$  is supported in  $C$ , we have:

$$\mathcal{F}_A^{-1}g = (\mathcal{F}_A^{-1}g) \left( \sum_{j=1}^n \chi_j \right) = \sum_{j=1}^n (\mathcal{F}_A^{-1}g) \chi_j.$$

Therefore  $g = \sum_{j=1}^n g_j$  with

$$g_j := \mathcal{F}_A((\mathcal{F}_A^{-1}g)\chi_j) \in \text{PW}(C(H_j) \cap C) \subset \text{PW}(U_\varepsilon(C)),$$

again by the classical Paley–Wiener theorem. Observe that  $U_\varepsilon(C)$  is compact, convex and  $W$ -invariant. So, by of Theorem 10.1,  $\mathcal{I}_\Pi g = \sum_{j=1}^n \mathcal{I}_\Pi g_j$  vanishes on  $A \setminus \exp U_\varepsilon(C)$ . Since  $\varepsilon > 0$  is arbitrary, we thus conclude that  $\text{supp } \mathcal{I}_\Pi g \subset C$ .  $\square$

The differential operator  $D_m$  allows a simple direct proof of Corollary 10.2 when  $m \in \mathcal{M}^+$  is even.

**Proof of Corollary 10.2 for  $m \in \mathcal{M}^+$  even.** Theorem 8.1 implies that  $\mathcal{F}_\Pi$  maps  $C_c^\infty(C)^W$  injectively into  $\text{PW}(C)^W$ . Lemma 9.4 proves that  $\mathcal{I}_\Pi$  maps  $\text{PW}(C)^W$  into  $C_c^\infty(C)^W$ . Finally, to show that  $\mathcal{F}_\Pi$  and  $k\mathcal{I}_\Pi$  are inverses to each other, see the proof of Theorem 1.3 in Section 15.  $\square$

## 11. The complex case

Before proceeding with the proof of the surjectivity of  $\mathcal{F}_\Theta$  in the general case, let us first treat the elementary situation corresponding to the complex case.

**Theorem 11.1.** *Let  $\Theta \subset \Pi$  be arbitrary, but suppose  $m_\alpha = 2$  for all  $\alpha \in \Sigma$ . Let  $C \subset \mathfrak{a}_\Theta$  be compact, convex and  $W_\Theta$ -invariant. Then  $\mathcal{I}_\Theta$  maps  $\text{PW}_\Theta(2; C)$  into  $C_c^\infty(C)^{W_\Theta}$ .*

**Proof.** Because of the definition of  $\mathcal{I}_\Theta$  and Lemma 9.3, we only need to prove that  $\text{supp } \mathcal{I}g \subset \exp W(C)$ . Indeed  $W(C) \cap \mathfrak{a}_\Theta = C$ .

Recall from Example 5.4 that  $\Delta(a)\pi(-\lambda)\varphi_\Pi(2; -\lambda, a) = \sum_{w \in W} \varepsilon(w)e^{-w\lambda(\log a)}$ . Therefore

$$\begin{aligned} \Delta(a)(\mathcal{I}g)(a) &= \int_{i\mathfrak{a}^*} g(\lambda)\pi(\lambda) \sum_{w \in W} \varepsilon(w)e^{-\lambda(\log w^{-1}a)} d\lambda \\ &= \sum_{w \in W} \varepsilon(w) \int_{i\mathfrak{a}^*} g(\lambda)\pi(\lambda)e^{-\lambda(\log w^{-1}a)} d\lambda \\ &= \sum_{w \in W} \varepsilon(w) [\mathcal{F}_A^{-1}(g \cdot \pi) \circ w^{-1}](a). \end{aligned}$$

Since  $\pi(\lambda)g(\lambda)$  is an entire function of exponential type  $C$  and rapidly decreasing, the classical Paley–Wiener theorem implies that  $\text{supp } \mathcal{F}_A^{-1}(g \cdot \pi) \subset \exp C$ . Thus  $\text{supp } \Delta\mathcal{I}g \subset \exp W(C)$ , which prove the result since  $\mathcal{I}g$  is smooth.  $\square$

## 12. Condition A

The proof of the surjectivity of the  $\Theta$ -hypergeometric transform will depend on the following condition on the pair  $(m, \Theta)$ .

**Condition A.** *Either*

$$m_\alpha \leq 2 \quad \text{for all } \alpha \in \Sigma^+ \setminus \langle \Theta \rangle^+, \quad (\text{A}_1)$$

*or there exists a  $\beta \in \Pi \setminus \Theta$  such that*

$$\Pi \setminus \Theta = \{\beta\} \quad \text{and} \quad \langle \beta, \alpha \rangle \geq 0 \quad \text{for all } \alpha \in \Sigma^+ \setminus \langle \Theta \rangle^+. \quad (\text{A}_2)$$

**Remark 12.1.** (1) The requirement  $m_\alpha \leq 2$  for all  $\alpha \in \Sigma^+ \setminus \langle \Theta \rangle^+$  means that there are at most singular  $\lambda$ -hyperplanes of the form  $\lambda_\alpha = 0$  with  $\alpha \in \Sigma^+ \setminus \langle \Theta \rangle^+$ . When all  $m_\alpha$  are equal, which happens for instance in the geometric case corresponding to  $K_\varepsilon$  symmetric spaces with even multiplicities, this requirement is equivalent to one of the following conditions:

- (a)  $\Theta = \Pi$ ,
- (b)  $m = 0$  (Euclidean case),
- (c)  $m = 2$  (complex case).

(2) In an irreducible reduced system  $\Sigma$  there are at most two root lengths. Moreover, if  $|\alpha| = |\beta|$ , then there is  $w \in W$  with  $w\alpha = \beta$  (see, e.g., [3], Proposition 11, p. 164). By  $W$ -invariance, multiplicity functions can therefore assume only two values on  $\Sigma$ .

Suppose  $\Sigma$  admits two root lengths, so  $\Sigma$  is of type  $B_l$  ( $l \geq 2$ ),  $C_l$  ( $l \geq 2$ ),  $F_4$  or  $G_2$ . It is then straightforward to verify that for  $|\Pi \setminus \Theta| = 1$  the set  $\Sigma^+ \setminus \langle \Theta \rangle^+$  always contains two root lengths. The condition  $m_\alpha \leq 2$  for all  $\alpha \in \Sigma^+ \setminus \langle \Theta \rangle^+$  is therefore equivalent to  $m_\alpha \leq 2$  for all  $\alpha \in \Sigma^+$ . We infer that, besides (a), (b) and (c) above, only the two following additional cases can occur:

- (d)  $m_\alpha = 0$  for all short  $\alpha \in \Sigma^+$  and  $m_\alpha = 2$  for all long  $\alpha \in \Sigma^+$ ,
- (e)  $m_\alpha = 0$  for all long  $\alpha \in \Sigma^+$  and  $m_\alpha = 2$  for all short  $\alpha \in \Sigma^+$ .

(3) The dual cone of  $\mathfrak{a}_\Theta$  is the  $W_\Theta$ -invariant cone:

$$\mathfrak{a}_\Theta^* := \{\lambda \in \mathfrak{a}^*: \lambda(H) \geq 0 \text{ for all } H \in \mathfrak{a}_\Theta\} = \sum_{\alpha \in \Sigma^+ \setminus \langle \Theta \rangle^+} \mathbb{R}_0^+ \alpha, \quad (56)$$

where  $\mathbb{R}_0^+$  denotes the set of nonnegative real numbers. Condition  $A_2$  requests that  $\mathfrak{a}_\Theta^*$  must not be “too wide” in  $\mathfrak{a}^*$ , or equivalently, that  $\mathfrak{a}_\Theta$  is “wide enough” in  $\mathfrak{a}$ . It is always satisfied in the following cases:

- (a)  $\Sigma$  of rank one;
- (b)  $\Sigma$  corresponds to a NCC space (not necessarily with even multiplicities) and  $\Pi \setminus \Theta = \{\gamma_1\}$ , where  $\gamma_1$  is the unique simple noncompact root.

The latter statement depends on the fact that  $\gamma_1 + \alpha$  is never a root when  $\alpha$  is a noncompact positive root because  $(\gamma_1 + \alpha)(X_0) = 2$  for  $X_0$  as in (6). Thus  $\langle \gamma_1, \alpha \rangle \geq 0$  by [3], Corollary to Theorem 1, p. 162.

The complete list of the cases in which Condition  $A_2$  is satisfied can be obtained using classification of the irreducible reduced root systems together with the property that  $\langle \alpha, \beta \rangle \geq 0$  for two non-proportional roots for which  $\alpha + \beta$  is not a root (see, e.g., [3], Chapter VI, Corollary to Theorem 1). For non-exceptional root systems, we get the following list, given in the notation of [3], Chapter VI, §4. If  $\Sigma$  is of type  $A_l$  ( $l \geq 1$ ), then Condition  $A_2$  holds for  $\Pi \setminus \Theta = \{\beta\}$  for every choice of  $\beta \in \Pi$ . The same is true for  $\Sigma$  of type  $B_2 = C_2$  or  $D_3$  or  $D_4$ . If  $\Sigma$  is of type  $B_l$  (with  $l > 2$ ) or  $C_l$  (with  $l > 2$ ) or  $D_l$  (with  $l > 4$ ), then Condition  $A_2$  holds only for  $\Pi \setminus \Theta = \{\alpha_l\}$ . Observe, though, that the root system  $G_2$  does not satisfy Condition  $A_2$  for any choice of the simple root  $\beta$ .

(4) Condition A is never satisfied by  $\Theta = \emptyset$  when  $\dim \mathfrak{a} > 1$ . Hence our Paley–Wiener theorem applies to the transform associated with Harish-Chandra series only in the even-multiplicity rank-one case. Notice however that the Harish-Chandra series agree with geometrically defined spherical functions only in the rank-one case.

(5) It is important to observe that, as a consequence of (1) and (3) above together with the list in Appendix C, all pairs  $(\Theta, m)$  corresponding to  $K_\varepsilon$  symmetric spaces with even multiplicities satisfy Condition A. Thus Condition A applies to all geometrical situations in which the theory of the  $\Theta$ -hypergeometric functions might be relevant.

In Sections 10 and 11 we have already given direct proofs of the Paley–Wiener theorem for  $\Theta = \Pi$  and in the complex case. The proof in the latter case can be easily modified

to apply also to the situation in which all multiplicities are either 0 or 2. Indeed, the shift operator  $D_2$  to be considered in this case is obtained as in Example 4.8 with the product over  $\Sigma^+$  replaced by a product over the positive roots with multiplicity 2. Thus all cases in which Condition  $A_1$  holds can be easily treated directly. For the proof of the surjectivity of the  $\Theta$ -hypergeometric transform onto the space  $PW_\Theta(m; C)$  we shall therefore restrict ourselves only to the case in which Condition  $A_2$  is satisfied. This simplifies some technical aspects of the argument. Nevertheless a unified proof for the pairs  $(\Theta, m)$  satisfying Condition  $A$  is possible, and is based on the observation in Corollary 4.14. We shall indicate in Remark 13.13 how to proceed in the general case.

### 13. Away from the walls of $\mathfrak{a}_\Theta$

In the following we assume  $\Theta \subsetneq \Pi$ . Set:

$$\begin{aligned} C_\Theta &:= \{H \in \mathfrak{a} : \alpha(H) = 0 \text{ for all } \alpha \in \Theta \text{ and } \alpha(H) > 0 \text{ for all } \alpha \in \Pi \setminus \Theta\} \\ &= \{H \in \mathfrak{a} : \alpha(H) = 0 \text{ for all } \alpha \in \langle \Theta \rangle \text{ and } \alpha(H) > 0 \text{ for all } \alpha \in \Sigma^+ \setminus \langle \Theta \rangle^+\}. \end{aligned} \quad (57)$$

Then  $C_\Theta$  is a closed convex subset of  $\overline{\mathfrak{a}^+}$  with  $\dim(\text{span}_{\mathbb{R}} C_\Theta) = |\Pi \setminus \Theta|$ . Moreover,  $\overline{\mathfrak{a}^+} = \bigcup_{\Theta \subset \Pi} C_\Theta$ . We refer to [17], pp. 25–26, for the proof of these statements. The group  $W_\Theta$  stabilizes  $C_\Theta$ . More precisely, the following lemma holds.

**Lemma 13.1** [17, Proposition 1.15]. *The following properties are equivalent for  $w \in W$ :*

- (a)  $w \in W_\Theta$ ,
- (b)  $wC_\Theta = C_\Theta$ ,
- (c)  $wH = H$  for all  $H \in C_\Theta$ .

Observe that  $C_\Pi = \{0\}$  and  $C_\emptyset = \mathfrak{a}^+$ . Let  $\mathbb{R}^+$  denote the set of positive real numbers. If  $|\Pi \setminus \Theta| = 1$ , then  $C_\Theta = \mathbb{R}^+ X^0$ , for some  $X^0 \in \mathfrak{a}_\Theta$ . In particular, if  $\Sigma$  is the root system of a NCC symmetric space and  $\Theta = \Pi_0$  is the set of compact positive simple roots, then  $C_\Theta = \mathbb{R}^+ X^0$ , where  $X^0 \in \mathfrak{a}$  is the cone generating element for the causal structure. Finally, observe from (57) that  $C_\Theta \subset \mathfrak{a}_\Theta$ .

We shall be interested in the closed cones in  $\mathfrak{a}_\Theta$ :

$$C(r, X^0) := rX^0 + \overline{\mathfrak{a}_\Theta}, \quad (58)$$

where  $X^0 \in C_\Theta$  and  $r > 0$ .

**Lemma 13.2.** *Let  $r > 0$  and  $X^0 \in C_\Theta$ . Then  $C(r, X^0)$  is a closed  $W_\Theta$ -invariant cone in  $\mathfrak{a}_\Theta$ . If  $a \in \mathfrak{a}^+ \setminus \exp C(r, X^0)$ , then there exists  $\beta \in \Pi \setminus \Theta$  with  $\beta(\log a - rX^0) < 0$ .*

**Proof.** The first statement follows immediately from (58) and Lemma 13.1.

Suppose now  $a \in A^+ \setminus \exp C(r, X^0)$ . Then

$$\log a - rX^0 \notin \overline{\mathfrak{a}_\Theta} = \{H \in \mathfrak{a}: \alpha(H) \geq 0 \text{ for all } \alpha \in \Sigma^+ \setminus \langle \Theta \rangle^+\}.$$

Hence, there exists a  $\beta_1 \in \Sigma^+ \setminus \langle \Theta \rangle^+$  with  $\beta_1(\log a - rX^0) < 0$ . Write

$$\beta_1 = \sum_{\alpha \in \Pi \setminus \Theta} n_\alpha \alpha + \sum_{\gamma \in \Theta} n_\gamma \gamma$$

with  $n_\alpha, n_\gamma \geq 0$ . Then  $(\Pi \setminus \Theta)_1 := \{\alpha \in \Pi \setminus \Theta: n_\alpha > 0\}$  is nonempty. Set:

$$\beta_2 := \sum_{\alpha \in (\Pi \setminus \Theta)_1} n_\alpha \alpha = \beta_1 - \sum_{\gamma \in \Theta} n_\gamma \gamma.$$

Then

$$\begin{aligned} \beta_2(\log a - rX^0) &= \beta_1(\log a - rX^0) - \sum_{\gamma \in \Theta} n_\gamma [\gamma(\log a) - r\gamma(X^0)] \\ &= \beta_1(\log a - rX^0) - \sum_{\gamma \in \Theta} n_\gamma \gamma(\log a) < 0. \end{aligned}$$

Since  $n_\alpha > 0$  for all  $\alpha \in (\Pi \setminus \Theta)_1$ , there must be  $\beta \in (\Pi \setminus \Theta)_1$  satisfying  $\beta(\log a - rX^0) < 0$ , as requested.  $\square$

Recall the definition of the dual cone  $\mathfrak{a}_\Theta^*$  of  $\mathfrak{a}_\Theta$  from (56). Its interior is

$$(\mathfrak{a}_\Theta^*)^0 = \{\lambda \in \mathfrak{a}^*: \lambda(H) > 0 \text{ for all } H \in \mathfrak{a}_\Theta\}.$$

**Lemma 13.3.** Suppose  $|\Pi \setminus \Theta| = 1$ . Then  $(\mathfrak{a}_\Theta^*)^0 \neq \emptyset$ .

**Proof.** Since  $|\Pi \setminus \Theta| = 1$  there is  $X^0 \neq 0$  so that  $C_\Theta = \mathbb{R}_+ X^0$ . We claim that  $\langle X^0, H \rangle > 0$  for all  $H \in \mathfrak{a}_\Theta$ . Indeed, suppose first  $H \in \overline{\mathfrak{a}_\Theta}$ . Then there is  $H_1 \in \overline{\mathfrak{a}^+}$  and  $w \in W_\Theta$  so that  $H = wH_1$ . By Lemma 13.1 and Lemma A in [5], p. 197, we have  $\langle X^0, H \rangle = \langle X^0, H_1 \rangle \geq 0$ . Hence  $\overline{\mathfrak{a}_\Theta} \subset \{H \in \mathfrak{a}: \langle X^0, H \rangle \geq 0\}$ , which implies  $\mathfrak{a}_\Theta \subset \{H \in \mathfrak{a}: \langle X^0, H \rangle > 0\}$ .

Let  $\lambda_0 \in \mathfrak{a}$  correspond to  $X^0$  under the identification of  $\mathfrak{a}$  and  $\mathfrak{a}^*$  under the inner product, i.e.,  $X^0 = A_{\lambda_0}$ . Then  $0 \neq \lambda_0 \in (\mathfrak{a}_\Theta^*)^0$ .  $\square$

For an even multiplicity function  $m \in \mathcal{M}^+$  we set:

$$\mathfrak{a}_\Theta^*(m) = \{\lambda \in \mathfrak{a}_\Theta^*: \lambda_\alpha \geq m_\alpha/2 \text{ for all } \alpha \in \Sigma^+ \setminus \langle \Theta \rangle^+\}. \quad (59)$$

The geometrical properties of  $\mathfrak{a}_\Theta^*(m)$  are collected in the following lemma:

**Lemma 13.4.** The set  $\mathfrak{a}_\Theta^*(m)$  is closed, convex and  $W_\Theta$ -invariant. If  $\mu \in \mathfrak{a}_\Theta^*(m)$  and  $t \geq 1$ , then also  $t\mu \in \mathfrak{a}_\Theta^*(m)$ . Furthermore, if  $|\Pi \setminus \Theta| = 1$ , then  $\mathfrak{a}_\Theta^*(m)$  has nonempty interior.

**Proof.** The only property needing some proof is that  $\mathfrak{a}_\Theta^*(m)$  has nonempty interior when  $|\Pi \setminus \Theta| = 1$ . Set:

$$(\mathfrak{a}_\Theta^*)^\dagger := \{\lambda \in \mathfrak{a}^*: \langle \lambda, \alpha \rangle > 0 \text{ for all } \alpha \in \Sigma^+ \setminus \langle \Theta \rangle^+\}.$$

Then it suffices to show that  $(\mathfrak{a}_\Theta^*)^\dagger \cap (\mathfrak{a}_\Theta^*)^0 \neq \emptyset$ . Indeed, if this is the case, then there is  $0 \neq \mu \in (\mathfrak{a}_\Theta^*)^\dagger \cap (\mathfrak{a}_\Theta^*)^0$ . If

$$t \geq \max_{\alpha \in \Sigma^+ \setminus \langle \Theta \rangle^+} \frac{m_\alpha}{2} \frac{\langle \alpha, \alpha \rangle}{\langle \mu, \alpha \rangle},$$

then  $t\mu \in (\mathfrak{a}_\Theta^*)^0$  and  $(t\mu)_\alpha = t \frac{\langle \mu, \alpha \rangle}{\langle \alpha, \alpha \rangle} \geq \frac{m_\alpha}{2}$ .

Recall the notation  $A_\lambda$  for the element in  $\mathfrak{a}$  identified with  $\lambda \in \mathfrak{a}^*$  via the inner product. In particular, as in the proof of Lemma 13.3, let  $A_{\lambda_0} = X^0$ . Observe that

$$\begin{aligned} (\mathfrak{a}_\Theta^*)^\dagger &= \{\lambda \in \mathfrak{a}^*: \langle \lambda, \nu \rangle > 0 \text{ for all } \nu \in \mathfrak{a}_\Theta^*\} \\ &= \{\lambda \in \mathfrak{a}^*: A_\lambda \in (\mathfrak{a}_\Theta^{**})^0 = \mathfrak{a}_\Theta\}. \end{aligned}$$

Thus, because of Lemma 13.3, we conclude  $\mathbb{R}^+ \lambda_0 \subset (\mathfrak{a}_\Theta^*)^\dagger \cap (\mathfrak{a}_\Theta^*)^0$ .  $\square$

The set  $\mathfrak{a}_\Theta^*(m)$  is introduced because it is a “large” closed subset of  $\mathfrak{a}_\Theta^*$  which is “away” from the possible singularities of every  $g \in \text{PW}_\Theta(m; C)$ . This is made precise by the following lemma:

**Lemma 13.5.** *Let  $C \subset \mathfrak{a}$  be compact and convex, and let  $g: \mathfrak{a}_\mathbb{C}^* \rightarrow \mathbb{C}$  satisfy  $e_\Theta^-(m; \lambda)g(\lambda) \in \text{PW}(C)$ . Then  $g$  is holomorphic on a neighborhood of the convex set  $i\mathfrak{a}^* - \mathfrak{a}_\Theta^*(m)$ . Furthermore, for every  $N \in \mathbb{N}$ , there is a constant  $C_N > 0$  such that for all  $\lambda \in i\mathfrak{a}^*$  and  $\mu \in -\mathfrak{a}_\Theta^*(m)$ ,*

$$|g(\lambda + \mu)| \leq C_N (1 + |\lambda|)^{-N} e^{qc(\mu)}. \quad (60)$$

**Proof.** The function  $g$  is holomorphic on the open set:

$$i\mathfrak{a}^* - \{\lambda \in \mathfrak{a}^*: \lambda_\alpha > m_\alpha/2 - 1 \text{ for all } \alpha \in \Sigma^+ \setminus \langle \Theta \rangle^+\}.$$

To prove the estimate, observe that, if  $\lambda \in i\mathfrak{a}^*$  and  $\mu \in -\mathfrak{a}_\Theta^*(m)$ , then

$$|e_\Theta^-(m; \lambda + \mu)| \geq \prod_{\alpha \in \Sigma^+ \setminus \langle \Theta \rangle^+} \prod_{k=-m_\alpha/2+1}^{m_\alpha/2-1} |\mu_\alpha - k| \geq 1. \quad \square$$

**Lemma 13.6.** *Let  $C \subset \mathfrak{a}_\Theta$  be compact, convex and  $W_\Theta$ -invariant. Assume that  $g : \mathfrak{a}_\mathbb{C}^* \rightarrow \mathbb{C}$  satisfy  $e_\Theta^-(m; \lambda)g(\lambda) \in PW(C)^{W_\Theta}$ . Then, for all  $\mu \in -\mathfrak{a}_\Theta^*(m)$  and  $a \in A$ ,*

$$\Delta(m; a) \int_{i\mathfrak{a}^*} g(\lambda) \varphi_\Pi(m; -\lambda, a) \frac{d\lambda}{|c_\Pi^+(m; \lambda)|^2} = \sum_{w \in W} D_m \int_{i\mathfrak{a}^*} g(\lambda + \mu) e^{-w(\lambda + \mu)(\log a)} d\lambda.$$

Moreover, for all  $a \in A$  and  $w \in W$ , the integral

$$\int_{i\mathfrak{a}^*} g(\lambda + \mu) e^{-w(\lambda + \mu)(\log a)} d\lambda \quad (61)$$

is independent of  $\mu \in -\mathfrak{a}_\Theta^*(m)$ . If  $w \in W_\Theta$ , then (61) is  $W_\Theta$ -invariant. In particular,

$$\int_{i\mathfrak{a}^*} g(\lambda + \mu) e^{-w(\lambda + \mu)(\log a)} d\lambda = \int_{i\mathfrak{a}^*} g(\lambda + \mu) e^{-(\lambda + \mu)(\log a)} d\lambda.$$

**Proof.** Recall formula (42) for the function  $\varphi_\Pi$ . Since

$$g(\lambda) \varphi_\Pi(m; -\lambda, a) |c_\Pi^+(m; \lambda)|^{-2} = g(\lambda) \pi(\lambda) e_\Pi^+(m; \lambda) \varphi_\Pi(m; -\lambda, a)$$

is entire and rapidly decreasing, we can apply Cauchy's theorem and get for all  $\mu \in -\mathfrak{a}_\Theta^*(m)$ :

$$\begin{aligned} \Delta(m; a) \int_{i\mathfrak{a}^*} g(\lambda) \varphi_\Pi(m; -\lambda, a) \frac{d\lambda}{|c_\Pi^+(m; \lambda)|^2} \\ &= \Delta(m; a) \int_{i\mathfrak{a}^*} g(\lambda + \mu) \pi(\lambda + \mu) e_\Pi^+(m; \lambda + \mu) \varphi_\Pi(m; -\lambda - \mu, a) d\lambda \\ &= \int_{i\mathfrak{a}^*} g(\lambda + \mu) \sum_{w \in W} D_m e^{-w(\lambda + \mu)(\log a)} d\lambda \\ &= \sum_{w \in W} D_m \int_{i\mathfrak{a}^*} g(\lambda + \mu) e^{-w(\lambda + \mu)(\log a)} d\lambda. \end{aligned}$$

The last equality is justified by the fact that  $D_m$  is a differential operator with smooth coefficients on  $A$  and by Lemma 13.5.

Set

$$F_w(a, \mu) := \int_{i\mathfrak{a}^*} g(\lambda + \mu) e^{-w(\lambda + \mu)(\log a)} d\lambda.$$

It is independent of  $\mu \in -\mathfrak{a}_\Theta^*(m)$  by Cauchy's theorem, since its integrand is holomorphic in a simply connected open neighborhood of  $i\mathfrak{a}^* - \mathfrak{a}_\Theta^*(m)$  on which an estimate of the form (60) holds. For  $w, v \in W_\Theta$ ,

$$\begin{aligned} F_w(va, \mu) &= \int_{i\mathfrak{a}^*} g(\lambda + \mu) e^{-v^{-1}w(\lambda + \mu)(\log a)} d\lambda \\ &= \int_{i\mathfrak{a}^*} g(w^{-1}vw\lambda + \mu) e^{-(w\lambda + v^{-1}w\mu)(\log a)} d\lambda \\ &= \int_{i\mathfrak{a}^*} g(\lambda + w^{-1}v^{-1}w\mu) e^{-(w\lambda + v^{-1}w\mu)(\log a)} d\lambda \quad (\text{by } W_\Theta\text{-invariance of } g) \\ &= F_w(a, w^{-1}v^{-1}w\mu) \\ &= F_w(a, \mu) \quad (\text{because } w^{-1}v^{-1}w\mu \in -\mathfrak{a}_\Theta^*(m)). \end{aligned}$$

Since  $F_w(a, \mu) = F_e(w^{-1}a, \mu)$ , it follows in particular that  $F_w(a, \mu) = F_e(a, \mu)$  for  $w \in W_\Theta$ .  $\square$

The possibility of separating points in  $\mathfrak{a}_\Theta$  from points outside  $\mathfrak{a}_\Theta$  is guaranteed by the following lemma. Recall from (50) the definition of  $W^\Theta$  and of the length function  $\ell$ .

**Lemma 13.7.** *If  $u \in W^\Theta \setminus \{e\}$ , then there is  $\beta \in \Pi \setminus \Theta$  with  $u\beta \in -\Sigma^+$ .*

**Proof.** If  $u \neq e$ , then  $\ell(u) > 1$ . Since  $\ell(u)$  represents the number of elements in  $\Sigma^+$  mapped by  $u$  into  $-\Sigma^+$ , there must be  $\beta \in \Pi$  so that  $u\beta \in -\Sigma^+$ . But this implies  $\ell(ur_\beta) < \ell(u)$  (see, e.g., [17], Lemma 1.6). Thus  $u\beta \notin \langle \Theta \rangle$  because of the definition of  $W^\Theta$ .  $\square$

In the following we assume that  $\Theta$  satisfies Condition  $A_2$ . This assumption is crucial because it will allow us either to separate points in  $\mathfrak{a}_\Theta$  from point outside it, or to move the contour of integration without encountering singular hyperplanes.

**Lemma 13.8.** *Suppose that Condition  $A_2$  is satisfied by  $\Theta$ , and let  $\beta \in \Pi \setminus \Theta$ . If  $\mu_0 \in \mathfrak{a}_\Theta^*(m)$ , then  $t\beta + \mu_0 \in \mathfrak{a}_\Theta^*(m)$  for all  $t \geq 0$ .*

**Proof.** Since  $\mathfrak{a}_\Theta^*$  is a closed cone, the assumption  $\beta, \mu_0 \in \mathfrak{a}_\Theta^*$  implies  $t\beta + \mu_0 \in \mathfrak{a}_\Theta^*$  for all  $t \geq 0$ . Consider now  $\alpha \in \Sigma^+ \setminus \langle \Theta \rangle^+$ . Then, assuming Condition  $A_2$ , we have:

$$(t\beta + \mu_0)_\alpha = t\beta_\alpha + (\mu_0)_\alpha \geq m_\alpha/2. \quad \square$$

**Lemma 13.9.** *Suppose Condition  $A_2$  is satisfied. Let  $a \in A^+$ ,  $\mu \in -\mathfrak{a}_\Theta^*(m)$ ,  $w \notin W_\Theta$ , and let  $g: \mathfrak{a}_\mathbb{C}^* \rightarrow \mathbb{C}$  satisfy  $e_\Theta^-(m; \lambda)g(\lambda) \in PW(C)^{W_\Theta}$ . Then*



$$\int_{i\mathfrak{a}^*} g(\lambda + \mu) e^{-w(\lambda + \mu)(\log a)} d\lambda = 0. \quad (62)$$

**Proof.** Write  $w = uv$  with  $v \in W_\Theta$  and  $u \in W^\Theta$ . If  $w \notin W_\Theta$ , then  $u \neq e$ . By Lemma 13.7, there is  $\beta \in \Pi \setminus \Theta$  so that  $\beta u \in -\Sigma^+$ . Let  $\mu_0 \in \mathfrak{a}_\Theta^*(m)$ . Then  $t\beta + \mu_0 \in \mathfrak{a}_\Theta^*(m)$  for all  $t \geq 1$  by Lemma 13.8. Set  $\mu_t := -(t\beta + \mu_0)$ . By  $W_\Theta$ -invariance of  $g$  and  $\mathfrak{a}_\Theta^*(m)$ , and since (61) is independent of  $\mu \in -\mathfrak{a}_\Theta^*(m)$ , we have:

$$\begin{aligned} \int_{i\mathfrak{a}^*} g(\lambda + \mu) e^{-w(\lambda + \mu)(\log a)} d\lambda &= \int_{i\mathfrak{a}^*} g(\lambda + \mu_t) e^{-w(\lambda + \mu_t)(\log a)} d\lambda \\ &= \int_{i\mathfrak{a}^*} g(v\lambda + v\mu_t) e^{-u(v\lambda + v\mu_t)(\log a)} d\lambda \\ &= \int_{i\mathfrak{a}^*} g(\lambda + v\mu_t) e^{-u(\lambda + v\mu_t)(\log a)} d\lambda \\ &= \int_{i\mathfrak{a}^*} g(\lambda + \mu_t) e^{-u(\lambda + \mu_t)(\log a)} d\lambda. \end{aligned}$$

Consider the last integral. By Lemma 13.5 we have for all  $N \in \mathbb{N}$ :

$$|g(\lambda + \mu_t)| \leq C_N (1 + |\lambda|)^{-N} e^{q_C(\mu_t)} = C_N e^{q_C(-\mu_0)} (1 + |\lambda|)^{-N} e^{tq_C(-\beta)}$$

with  $q_C(-\beta) \leq 0$  because  $\beta \in \Pi \setminus \Theta$  and  $C \subset \mathfrak{a}_\Theta$ . Moreover

$$|e^{-u(\lambda + \mu_t)(\log a)}| = |e^{-u\mu_t(\log a)}| = e^{u\mu_0(\log a)} e^{tu\beta(\log a)}.$$

with  $u\beta(\log a) < 0$  because  $u\beta \in -\Sigma^+$  and  $a \in A^+$ . The result then follows by taking the limit  $t \rightarrow +\infty$ .  $\square$

**Theorem 13.10.** Suppose that Condition A<sub>2</sub> is satisfied. Let  $C \subset \mathfrak{a}_\Theta$  be compact and convex, and let  $g : \mathfrak{a}_\mathbb{C}^* \rightarrow \mathbb{C}$  satisfy  $e_\Theta^-(m; \lambda)g(\lambda) \in \text{PW}(C)^{W_\Theta}$ . Suppose furthermore that  $\mu \in -\mathfrak{a}_\Theta^*(m)$ . Then for all  $a \in A_\Theta$ :

$$\begin{aligned} \Delta(m; a) \int_{i\mathfrak{a}^*} g(\lambda) \varphi_\Pi(m; -\lambda, a) \frac{d\lambda}{|c_\Pi^+(m; \lambda)|^2} \\ = |W_\Theta| D_m \int_{i\mathfrak{a}^*} g(\lambda + \mu) e^{-(\lambda + \mu)(\log a)} d\lambda. \end{aligned} \quad (63)$$

**Proof.** Let  $a \in A^+$ . By Lemmas 13.6 and 13.9,

$$\begin{aligned}
\Delta(m; a) \int_{i\mathfrak{a}^*} g(\lambda) \varphi_{\Pi}(m; -\lambda, a) \frac{d\lambda}{|c_{\Pi}^+(m; \lambda)|^2} &= \sum_{w \in W} D_m \int_{i\mathfrak{a}^*} g(\lambda + \mu) e^{-w(\lambda + \mu)(\log a)} d\lambda \\
&= \sum_{w \in W_{\Theta}} D_m \int_{i\mathfrak{a}^*} g(\lambda + \mu) e^{-w(\lambda + \mu)(\log a)} d\lambda \\
&= |W_{\Theta}| D_m \int_{i\mathfrak{a}^*} g(\lambda + \mu) e^{-(\lambda + \mu)(\log a)} d\lambda.
\end{aligned}$$

Both sides of (63) are  $W_{\Theta}$ -invariant and continuous. Hence they agree on the entire  $A_{\Theta}$ .  $\square$

As an immediate consequence we obtain an alternative expression for the inversion formula for the  $\Theta$ -hypergeometric transform. It is a modification by the differential operator  $D_m$  of the inversion formula for the Euclidean Fourier transform.

**Corollary 13.11.** *Suppose that Condition  $A_2$  is satisfied. Let  $f \in C_c^{\infty}(C)^{W_{\Theta}}$ . Then for all  $a \in A_{\Theta}$  and  $\mu \in -\mathfrak{a}_{\Theta}^*(m)$ ,*

$$\Delta(m; a) f(a) = (-1)^{d(\Theta, m)} k |W|^2 D_m \int_{i\mathfrak{a}^*} (\mathcal{F}_{\Theta} f)(m; \lambda + \mu) e^{-(\lambda + \mu)(\log a)} d\lambda, \quad (64)$$

where  $k$  is the normalizing constant in Theorem 7.3 and  $d(\Theta, m)$  is as in (17).

**Proof.** Immediate from Theorems 7.3 and 13.10.  $\square$

Recall the notation  $C(r, X^0) = rX^0 + \overline{\mathfrak{a}_{\Theta}}$  from (58).

**Lemma 13.12.** *Suppose Condition  $A_2$  is satisfied. Let  $r > 0$  and  $X^0 \in C_{\Theta}$ . Let  $C \subset \mathfrak{a}_{\Theta}$  be compact and convex, and let  $g : \mathfrak{a}_{\mathbb{C}}^* \rightarrow \mathbb{C}$  satisfy  $e_{\Theta}^-(m; \lambda) g(\lambda) \in PW(C)^{W_{\Theta}}$ . Assume furthermore that  $C \subset C(r, X^0)$ . Then  $\text{supp } \mathcal{I}_{\Theta} g \subset C(r, X^0)$ .*

**Proof.** Let  $a \in A^+ \setminus \exp C(r, X^0)$ , and let  $\beta \in \Pi \setminus \Theta$  be as in Lemma 13.2. For a fixed  $\mu_0 \in \mathfrak{a}_{\Theta}^*(m)$  and  $t \geq 1$  we set  $\mu_t := -(t\beta + \mu_0)$ . As in Lemma 13.9, we have  $\mu_t \in -\mathfrak{a}_{\Theta}^*(m)$  and, for all  $N \in \mathbb{N}$ ,

$$|g(\lambda + \mu_t)| e^{-(\lambda + \mu_t)(\log a)} \leq C_N e^{q_C(-\mu_0)} e^{\mu_0(\log a)} (1 + |\lambda|)^{-N} e^{tq_C(-\beta)} e^{t\beta(\log a)}.$$

Notice that  $C \subset C(r, X^0)$  implies  $C = rX^0 + C'$ , where  $C' := C - rX^0 \subset \overline{\mathfrak{a}_{\Theta}^+}$  is compact, convex,  $W_{\Theta}$ -invariant. Hence

$$q_C(-\beta) = \sup_{H \in C} [-\beta(H)] = -\beta(rX^0) + \sup_{H \in C'} [-\beta(H)] = -\beta(rX^0) + q_{C'}(-\beta)$$

with  $q_{C'}(-\beta) \leq 0$  because  $\beta \in \Pi \setminus \Theta$ . Thus

$$|g(\lambda + \mu_t)| e^{-(\lambda + \mu_t)(\log a)} \leq C_N e^{\mu_0(\log a)} (1 + |\lambda|)^{-N} e^{t\beta(\log a - rX^0)},$$

which converges to 0 as  $t \rightarrow +\infty$  by Lemma 13.2. Together with Theorem 13.10, this proves that for  $a \in A^+ \setminus \exp C(r, X^0)$  we have  $\Delta(m; a)(\mathcal{I}g)(a) = 0$ . Since  $\mathcal{I}g$  is  $W_\Theta$ -invariant and smooth on  $A$  by Lemma 9.3, we conclude that  $\mathcal{I}g(a) = 0$  for  $a \in A \setminus \bigcup_{w \in W} \exp w(C(r, X^0)) = A \setminus \bigcup_{u \in W_\Theta} \exp u(C(r, X^0))$ . Because of Lemma 6.3, the sets  $u(C(r, X^0))$  are pairwise disjoint. Thus  $\text{supp } \mathcal{I}_\Theta g \subset C(r, X^0)$ .  $\square$

**Remark 13.13.** One could use Corollary 4.17 and replace the set  $\mathfrak{a}_\Theta^*(m)$  with

$$\{\lambda \in \mathfrak{a}_\Theta^*: \lambda_\alpha \geq m_\alpha/2 \text{ for all } \alpha \in \Sigma^+ \setminus \langle \Theta \rangle^+ \text{ with } m_\alpha > 2\}.$$

This would allow us a proof of the previous lemmas which applies also to the complex case (which generally does not satisfy Condition A<sub>2</sub>).

The results obtained so far in this section apply in particular to every element  $g \in \text{PW}_\Theta(m; C)$ , but the additional condition that  $\text{P}_\Theta^{\text{av}} g$  extends to be entire on  $\mathfrak{a}_\mathbb{C}^*$  has not been used. This condition is needed in the proof of the Paley–Wiener theorem to apply Lemma 13.12, in particular to know that  $\mathcal{I}_\Theta g$  has compact support.

At this point we can already prove that  $\mathcal{I}_\Theta$  maps  $\text{PW}_\Theta(m; C)$  into  $C_c^\infty(C)^{W_\Theta}$ , but only for a special class of compact convex and  $W_\Theta$ -invariant exhausting subsets  $C$  of  $\mathfrak{a}_\Theta$ .

Recall for  $H \in \mathfrak{a}$  the notation  $C(H) := \text{conv}(W(H))$ .

**Corollary 13.14.** *Let  $C \subset \mathfrak{a}_\Theta$  be a compact, convex and  $W_\Theta$ -invariant subset of the form  $C = C(r, X^0) \cap C(H)$  with  $r > 0$ ,  $X^0 \in C_\Theta$  and  $H \in \mathfrak{a}$ . Suppose  $g \in \text{PW}_\Theta(m; C)$ . Assume Condition A<sub>2</sub> is satisfied. Then  $\text{supp } \mathcal{I}_\Theta g \subset \exp C$ .*

**Proof.** Lemmas 9.3 and 13.12 ensure that  $\mathcal{I}_\Theta g$  is smooth and compactly supported inside  $C(H)$ . Moreover, Lemma 13.12 guarantees that  $\text{supp } \mathcal{I}_\Theta g \subset C(r, X^0)$ .  $\square$

## 14. Support properties

By means of Condition A<sub>2</sub> we could prove in Section 13 that  $\mathcal{I}_\Theta g$  is compactly supported inside  $A_\Theta$  for every  $g \in \text{PW}_\Theta(m; C)$ , where  $C$  is an arbitrary compact convex and  $W_\Theta$ -invariant subset of  $\mathfrak{a}_\Theta$ . In this section we apply a support theorem proven in [24] to show that the support of  $\mathcal{I}_\Theta g$  is indeed contained in  $\exp C$ .

Recall the  $W$ -invariant polynomial  $q$  introduced in (52). It satisfies  $q(-\lambda) = q(\lambda)$  for all  $\lambda \in \mathfrak{a}_\mathbb{C}^*$ , and  $qg \in \text{PW}(C)^{W_\Theta}$  for all  $g \in \text{PW}_\Theta(m; C)$ . Since  $q \in S(\mathfrak{a}_\mathbb{C})^W$ , we can consider the corresponding operator  $D(m; q) \in \mathbb{D}(\mathfrak{a}, \Sigma, m)$ , that is

$$D(m; q) = \Upsilon \left( \prod_{\alpha \in \Sigma} \prod_{k=-m_\alpha/2+1}^{m_\alpha/2-1} (T(m; H_\alpha/2) - k) \right) \quad (65)$$

(see Definition 2.2 and (10)). Hence

$$D(m; q)\varphi_{\Theta}(m; \lambda, a) = q(\lambda)\varphi_{\Theta}(m; \lambda, a) \quad (66)$$

for all  $a \in A^+$  and all  $\Theta \subset \Pi$ . Moreover, there is  $k \in \mathbb{N}$  such that

$$D_q := \Delta^k D(m; q) \quad (67)$$

is a  $W$ -invariant differential operator on  $A$  with analytic coefficients.

**Lemma 14.1.** *Suppose  $g \in \text{PW}_{\Theta}(m, C)$ . Then, for all  $a \in A_{\Theta}$ ,*

$$D_q(\mathcal{I}_{\Theta}g)(a) = |W_{\Theta}| \Delta(k - m; a) D_m \mathcal{F}_A^{-1}(qg)(a) \quad (68)$$

with  $\Delta(k - m; a) D_m \mathcal{F}_A^{-1}(qg) \in C_c^{\infty}(C)$ .

**Proof.** Because of (66) and Theorem 13.10, for all  $a \in A$  we have:

$$\begin{aligned} \Delta(m; a) D_q(\mathcal{I}_{\Theta}g)(a) &= \Delta(m; a) \Delta^k(a) \int_{i\mathfrak{a}^*} g(\lambda) q(\lambda) \varphi_{\Pi}(m; -\lambda, a) \frac{d\lambda}{|c_{\Pi}^+(m; -\lambda)|^2} \\ &= |W_{\Theta}| \Delta^k(a) D_m \int_{i\mathfrak{a}^*} q(\lambda + \mu) g(\lambda + \mu) e^{-(\lambda + \mu)(\log a)} d\lambda. \end{aligned}$$

Since  $qg \in \text{PW}(C)$ , we can shift the contour of integration and obtain:

$$\begin{aligned} \Delta(m; a) D_q(\mathcal{I}_{\Theta}g)(a) &= |W_{\Theta}| \Delta^k(a) D_m \int_{i\mathfrak{a}^*} q(\lambda) g(\lambda) e^{-\lambda(\log a)} d\lambda \\ &= |W_{\Theta}| \Delta^k(a) D_m \mathcal{F}_A^{-1}(qg)(a). \end{aligned} \quad (69)$$

The classical Paley–Wiener theorem guarantees that  $\text{supp } \mathcal{F}_{\Theta}^{-1}(qg) \subset C$ . Since  $D_q(\mathcal{I}_{\Theta}g)$  is smooth, we conclude that  $\Delta(m; a)$  must divide the right-hand side of (69). Thus

$$\Delta(m - k) D_m \mathcal{F}_A^{-1}(qg) = \frac{\Delta(m) D_m \mathcal{F}_A^{-1}(qg)}{\Delta^k(a)}$$

is smooth and supported in  $C$ .  $\square$

Lemma 14.1 shows that  $D_q(\mathcal{I}_{\Theta}g)$  is supported inside  $\exp C$ . We can therefore conclude that  $\mathcal{I}_{\Theta}$  maps  $\text{PW}_{\Theta}(m; C)$  to  $C_c^{\infty}(C)^{W_{\Theta}}$  if we prove that the support of  $\mathcal{I}_{\Theta}g$  is contained in  $\exp C$  when the support of  $D_q(\mathcal{I}_{\Theta}g)$  is in  $\exp C$ . This is guaranteed by the following proposition, which depends on the particular form of the principal symbol of the differential operator  $D_q$ .

**Proposition 14.2.** *Let  $D_q$  be the differential operator from (67). Then*

$$\operatorname{supp} f \subset \exp C \iff \operatorname{supp} D_q f \subset \exp C$$

for every  $f \in C_c^\infty(A)^{W_\Theta}$  and for every compact, convex and  $W_\Theta$ -invariant subset  $C \subset \mathfrak{a}_\Theta$  with nonempty interior.

**Proof.** The implication of  $\operatorname{supp} D_q f \subset \exp C$  from  $\operatorname{supp} f \subset \exp C$  is obvious. The converse inclusion follows from Theorem 1.2 in [24]. Indeed, the highest homogeneous part of  $q(\lambda)$  is:

$$q_h(\lambda) := \prod_{\alpha \in \Sigma} \lambda_{w\alpha}^{m_\alpha - 1},$$

and Formula (65) implies that the highest homogeneous part of  $D(m; q)$  is:

$$\prod_{\alpha \in \Sigma} (\partial(H_{w\alpha}/2))^{m_\alpha - 1}.$$

The principal symbol of  $D_q$  at  $(H, \lambda) \in \mathfrak{a} \times \mathfrak{a}^*$  is therefore

$$\sigma(D_q)(H, \lambda) = \Delta^k(H) \prod_{\alpha \in \Sigma} \lambda_{w\alpha}^{m_\alpha - 1}. \quad \square \quad (70)$$

**Theorem 14.3.** *Suppose  $\Theta$  satisfies Condition A<sub>2</sub>. Let  $C \subset \mathfrak{a}_\Theta$  be an arbitrary compact, convex and  $W_\Theta$ -invariant subset. Then  $\operatorname{supp} \mathcal{I}_\Theta g \subset C$  for all  $g \in \operatorname{PW}_\Theta(m; C)$ .*

**Proof.** Immediate consequence of Lemma 14.1 and Proposition 14.2.  $\square$

## 15. Conclusion of the proof of the Paley–Wiener theorem

*Conclusion of the proof of Theorem 1.3.* Because of Theorems 8.1 and 14.3, we only need to prove that  $\mathcal{F}_\Theta : C_c^\infty(C)^{W_\Theta} \rightarrow \operatorname{PW}_\Theta(m; C)$  and  $\mathcal{I}_\Theta : \operatorname{PW}_\Theta(m; C) \rightarrow C_c^\infty(C)^{W_\Theta}$  are (up to a fixed constant depending on  $\Theta$  and  $m$ ) inverse to each other.

Let  $g \in \operatorname{PW}_\Theta(m; C)$ . We have then proven that  $\mathcal{F}_\Theta \mathcal{I}_\Theta g \in \operatorname{PW}_\Theta(m; C)$ . Observe that  $\mathcal{I}_\Pi g = \mathcal{I}g$  is the  $W$ -invariant extension of  $\mathcal{I}_\Theta g$  to  $A$ . Corollaries 7.2 and 10.2 together with Remark 9.2 therefore yield

$$(\mathcal{P}_\Theta^{\operatorname{av}} \mathcal{F}_\Theta \mathcal{I}_\Theta g)(m; \lambda) = (-1)^{d(\Theta, m)} (\mathcal{F}_\Pi \mathcal{I}_\Pi g)(m; \lambda) = \frac{1}{k(\Theta, m)} \mathcal{P}_\Theta^{\operatorname{av}} g(m; \lambda),$$

where we have set:

$$k(\Theta, m) := (-1)^{d(\Theta, m)} k \frac{|W_\Theta|}{|W|}.$$

The injectivity of  $P_{\Theta}^{\text{av}}$  from Proposition 6.5 gives then

$$\mathcal{F}_{\Theta} \mathcal{I}_{\Theta} g = \frac{1}{k(\Theta, m)} g$$

for all  $g \in \text{PW}_{\Theta}(m; C)$ . Conversely, the inversion formula of Theorem 7.3 yields

$$k(\Theta, m) \mathcal{I}_{\Theta} \mathcal{F}_{\Theta} f = f$$

for  $f \in C_c^{\infty}(C)^{W_{\Theta}}$ . This concludes the proof of Theorem 1.3.

## Appendix A. Estimates for the Harish-Chandra series

In this appendix we prove estimates for the derivatives of the Harish-Chandra series in the spectral parameter. As a corollary, they will provide a proof for Lemma 4.13. In the case, when no differentiation occurs, these estimates have been proven in [32]. The general case follows by a straightforward modification of the arguments from the special case. We therefore only outline the proof and refer the reader to Section 4 in [32] for the details.

In the following we adopt the notation of Sections 2 and 3, but we allow  $\Sigma$  to be an arbitrary (not necessarily reduced) root system. We also assume that  $m_{\alpha} = 0$  is equivalent to  $\alpha \notin \Sigma$ . As in Section 4, we denote by  $(\lambda_1, \dots, \lambda_l)$  the complex coordinates in  $\mathfrak{a}_{\mathbb{C}}^*$  associated with an orthonormal basis  $(\xi_1, \dots, \xi_l)$  of  $\mathfrak{a}^*$ . Moreover, if  $I = (\iota_1, \dots, \iota_l) \in \mathbb{N}_0^l$  is a multi-index and  $H \in \mathfrak{a}_{\mathbb{C}}$ , we set:

$$\xi^I(H) := (\xi_1(H))^{\iota_1} \cdots (\xi_l(H))^{\iota_l}.$$

The product rule for the derivation is then given by:

$$\partial_{\lambda}^I(fg) = \sum_{J+K=I} \frac{I!}{J!K!} (\partial_{\lambda}^J f)(\partial_{\lambda}^K g).$$

**Lemma A.1.** *Suppose  $I = (\iota_1, \dots, \iota_l)$ ,  $p \in S(\mathfrak{a}_{\mathbb{C}})$ ,  $m \in \mathbb{N}$ ,  $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$  and  $a \in A$ . Then*

- (a)  $\partial_{\lambda}^I e^{\lambda(\log a)} = \xi^I(\log a) e^{\lambda(\log a)}$ .
- (b) *Let  $a$  be a meromorphic function on a domain  $D \subset \mathfrak{a}_{\mathbb{C}}^*$ . Suppose  $p(\lambda)a(\lambda)$  is holomorphic on  $D$ . Then also*

$$p(\lambda)^{|I|+1} \partial_{\lambda}^I a(\lambda)$$

*is holomorphic in  $D$ .*

Estimates for the Harish-Chandra series and its derivatives will be obtained as in [7] and [32] using the modified Harish-Chandra series

$$\Psi(m; \lambda, a) := \Delta(m/2; a) \Phi(m; \lambda, a)$$

with  $\Delta(m)$  defined by (9). In particular, for every multi-index  $I$ , we have:

$$\partial_\lambda^I \Psi(m; \lambda, a) := \Delta(m/2; a) \partial_\lambda^I \Phi(m; \lambda, a). \quad (\text{A.1})$$

For  $S > 0$ , set  $\mathfrak{a}^+(S) := \{H \in \mathfrak{a}: \alpha_j(H) > S \text{ for all } j = 1, \dots, l\}$  and  $A^+(S) := \exp \mathfrak{a}^+(S)$ . Suppose  $\lambda \in \mathfrak{a}_\mathbb{C}^*$  satisfies  $\langle \mu, \mu - 2\lambda \rangle \neq 0$  for all  $\mu \in 2\Lambda \setminus \{0\}$ . Then, on  $A^+$ , we have the series expansions (15) and

$$\begin{aligned} \Delta(m/2; a) &= e^{\rho(m)(\log a)} \prod_{\alpha \in \Sigma^+} (1 - e^{-2\alpha(\log a)})^{m_\alpha/2} \\ &= e^{\rho(m)(\log a)} \sum_{\mu \in 2\Lambda} b_\mu(m) e^{-\mu(\log a)} \end{aligned} \quad (\text{A.2})$$

with  $b_0(m) = 1$ . The Cauchy product of (A.2) and (15) yields the series expansion,

$$\Psi(m; \lambda, a) = e^{\lambda(\log a)} \sum_{\mu \in 2\Lambda} a_\mu(m; \lambda) e^{-\mu(\log a)}, \quad a \in A^+, \quad (\text{A.3})$$

with

$$a_\mu(m; \lambda) := \sum_{\substack{v, \eta \in 2\Lambda \\ v + \eta = \mu}} b_v(m) \Gamma_\eta(m; \lambda) \quad (\text{A.4})$$

and  $a_0(m; \lambda) = 1$ . The series converges absolutely in  $A^+$  and uniformly in  $A^+(S)$ .

The coefficients  $a_\mu(m; \lambda)$  satisfy the recursion relations:

$$\begin{aligned} a_\mu(m; \lambda) \langle \mu - 2\lambda, \mu \rangle &= \sum_{\alpha \in \Sigma^+} m_\alpha (2 - m_\alpha - 2m_{2\alpha}) \langle \alpha, \alpha \rangle \\ &\quad \times \sum_{\substack{k \in \mathbb{N} \\ \mu - 2k\alpha \in 2\Lambda}} k a_{\mu - 2k\alpha}(m; \lambda) \end{aligned} \quad (\text{A.5})$$

for  $\mu \in 2\Lambda \setminus \{0\}$ , with initial condition  $a_0(m; \lambda) = 1$ . See Eq. (37) in [32]. It follows in particular that for  $\mu \in 2\Lambda \setminus \{0\}$  each coefficient  $a_\mu(m; \lambda)$  is a rational function of  $\lambda \in \mathfrak{a}_\mathbb{C}^*$  with at most simple poles along the hyperplanes

$$\mathcal{H}_\eta := \{\lambda \in \mathfrak{a}_\mathbb{C}^*: \langle \eta - 2\lambda, \eta \rangle = 0\}$$

for some  $\eta \in 2\Lambda \setminus \{0\}$  with  $\eta \leq \mu$ .

In the following  $R$  will always denote a *finite* positive real number. We define:

$$\mathfrak{a}_{\mathbb{C}}^*(R) := \{\lambda \in \mathfrak{a}_{\mathbb{C}}^*: \operatorname{Re}\langle \lambda, \alpha \rangle < R \text{ for all } \alpha \in \Sigma^+\}, \quad (\text{A.6})$$

$$\mathcal{X}_R := \{\eta \in 2\Lambda \setminus \{0\}: \mathcal{H}_{\eta} \cap \mathfrak{a}_{\mathbb{C}}^*(2R) \neq \emptyset\}, \quad (\text{A.7})$$

$$p_R(\lambda) := \prod_{\eta \in \mathcal{X}_R} \langle \eta - 2\lambda, \eta \rangle. \quad (\text{A.8})$$

By Lemma 4.4 in [32], the set  $\mathcal{X}_R$  is finite, and there is  $\omega > 0$  such that  $\mathcal{X}_R = \emptyset$  for  $0 < R \leq \omega$ . We convene that  $p_R(\lambda) \equiv 1$  when  $\mathcal{X}_R = \emptyset$  and denote by  $\deg p_R$  the degree of the polynomial  $p_R$ .

**Lemma A.2.** *Let  $R, S > 0$  be arbitrarily fixed. Then the series*

$$p_R(\lambda) \sum_{\mu \in 2\Lambda} a_{\mu}(m; \lambda) e^{-\mu(\log a)} \quad (\text{A.9})$$

*converges uniformly in  $(\lambda, a) \in \overline{V} \times A^+(S)T$ , where  $V$  is any open subset with compact closure  $\overline{V} \subset \mathfrak{a}_{\mathbb{C}}^*(2R)$ . Consequently,*

$$p_R(\lambda) \Psi(m; \lambda, a) = p_R(\lambda) e^{\lambda(\log a)} \sum_{\mu \in 2\Lambda} a_{\mu}(m; \lambda) e^{-\mu(\log a)}$$

*and, for every multi-index  $I$ ,*

$$p_R(\lambda)^{|I|+1} \partial_{\lambda}^I \Psi(m; \lambda, a)$$

*are holomorphic functions of  $(\lambda, a) \in \mathfrak{a}_{\mathbb{C}}^*(2R) \times A^+(S)U$ , where  $U \subset T$  is a neighborhood of  $e$  on which  $e^{\lambda(\log a)}$  is holomorphic for all  $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ .*

**Proof.** The first part of the lemma depends on the fact that  $p_R(\lambda) \Gamma_{\mu}(m; \lambda)$  is holomorphic in  $\mathfrak{a}_{\mathbb{C}}^*(2R)$  for all  $\eta \in 2\Lambda$ , together with (A.4) and an easy modification of Lemma 2.1 in [26]. The final statement follows from Lemma A.1(b).  $\square$

Because of Lemma A.2, the series (A.3) converges uniformly on compact subsets of  $(\mathfrak{a}_{\mathbb{C}}^* \setminus (\bigcup_{\mu \in 2\Lambda \setminus \{0\}} \mathcal{H}_{\mu})) \times A^+U$ . Termwise differentiation and Lemma A.1(a) yield there:

$$\begin{aligned} \partial_{\lambda}^I \Psi(m; \lambda, a) &= \sum_{\mu \in 2\Lambda} \partial_{\lambda}^I (e^{\lambda(\log a)} a_{\mu}(m; \lambda)) e^{\mu(\log a)} \\ &= e^{\lambda(\log a)} \sum_{\mu \in 2\Lambda} \sum_{J+K=I} \frac{I!}{J!K!} \xi^J(\log a) \partial_{\lambda}^K a_{\mu}(m; \lambda) e^{-\mu(\log a)}. \end{aligned} \quad (\text{A.10})$$

Finding estimates for  $\partial_{\lambda}^I \Psi(m; \lambda, a)$  on  $A^+$  is therefore reduced to estimating the  $\partial_{\lambda}^K a_{\mu}(m; \lambda)$ .



**Lemma A.3.** Suppose  $R > 0$  and  $\mu \in 2\Lambda$ . Then the following holds for every multi-index  $I$ :

- (a)  $p_R^{|I|+1}(\lambda) \partial_\lambda^I a_\mu(m; \lambda)$  is holomorphic in  $\lambda \in \mathfrak{a}_\mathbb{C}^*(2R)$ .  
 (b) For all  $\nu < \mu$ , the function  $\frac{p_R(\lambda)^{|I|+1} \partial_\lambda^I a_\nu(m; \lambda)}{\langle \mu - 2\lambda, \mu \rangle}$  is holomorphic in  $\lambda \in \mathfrak{a}_\mathbb{C}^*(2R)$ .

**Proof.** This follows from Lemma 4.5 in [32] and Lemma A.1(b).  $\square$

Let  $e_j \in \mathbb{N}_0^l$  denote the multi-index with all coordinates zero except for the  $j$ th equal to 1. Differentiation of (A.5) yields for multi-indices  $I = (\iota_1, \dots, \iota_l) \neq 0$  the recursion relations:

$$\begin{aligned} \langle \mu - 2\lambda, \mu \rangle \partial_\lambda^I a_\mu(m; \lambda) &= 2 \sum_{\substack{j \in \{1, \dots, l\} \\ I - e_j \in \mathbb{N}_0^l}} \langle \xi_j, \mu \rangle \iota_j \partial_\lambda^{I - e_j} a_\mu(m; \lambda) \\ &+ \sum_{\alpha \in \Sigma^+} m_\alpha (2 - m_\alpha - 2m_{2\alpha}) \langle \alpha, \alpha \rangle \\ &\times \sum_{\substack{k \in \mathbb{N} \\ \mu - 2k\alpha \in 2\Lambda}} k \partial_\lambda^I a_{\mu - 2k\alpha}(m; \lambda) \end{aligned} \quad (\text{A.11})$$

for  $\mu \in 2\Lambda \setminus \{0\}$ , with initial condition  $\partial_\lambda^I a_0(m; \lambda) = 0$ .

The procedure for estimating the derivatives  $\partial_\lambda^I a_\mu(m; \lambda)$  is based, as in the case  $I = 0$  of [32], on the comparison of the recursion relations (A.11) with the recursion relations satisfied by the coefficients  $d_\mu(m; c)$  of the series expansion of the function:

$$\Delta_c(m) := e^{c\rho(m)} \Delta(-cm/2) = \prod_{\alpha \in \Sigma^+} (1 - e^{-2\alpha})^{-cm_\alpha/2}, \quad (\text{A.12})$$

where  $c \in [0, 1)$  will be suitably chosen.

**Lemma A.4.** Suppose  $m \in \mathcal{M}^+$  and  $c \in \mathbb{R}$ .

- (a) [32, Lemma 4.8]. The function  $\Delta_c(m)$  defined in (A.12) admits the series expansion

$$\Delta_c(m) = \sum_{\mu \in 2\Lambda} d_\mu(m; c) e^{-\mu} \quad (\text{A.13})$$

which converges absolutely in  $A^+$  and uniformly in  $A^+(S)$  for all  $S > 0$ . We have  $d_0(m; c) = 1$ , and, if we assume  $c \in (0, \infty)$ , then  $d_\mu(m; c) > 0$  for all  $\mu \in 2\Lambda$ .

- (b) [32, Lemma 4.10]. Suppose  $H \in \mathfrak{a}$  and  $c \in [0, \infty)$ . Then the coefficients  $d_\mu(m; c)$  of the series (A.13) satisfy the recurrence relations:

$$\begin{aligned}
(\langle \mu, \mu \rangle + \mu(H))d_\mu(m; c) &= \sum_{\alpha \in \Sigma^+} \sum_{\substack{k \in \mathbb{N} \\ \mu - 2k\alpha \in 2\Lambda}} \left[ 2cm_\alpha \left( \frac{\alpha(H)}{2} - c\langle \rho(m), \alpha \rangle \right) \right. \\
&\quad \left. + kcm_\alpha (cm_\alpha + 2cm_{2\alpha} + 2)\langle \alpha, \alpha \rangle \right] d_{\mu-2k\alpha}(m; c)
\end{aligned} \tag{A.14}$$

for  $\mu \in 2\Lambda \setminus \{0\}$ , and  $d_0(m; c) = 1$ .

The constant  $c$  is chosen according to the following lemma:

**Lemma A.5** [32, Lemma 4.11]. *There exist constants  $0 \leq c < 1$  and  $r > 1$  such that the inequality*

$$c(cm_\alpha + 2cm_{2\alpha} + 2) \geq r|m_\alpha + 2m_{2\alpha} - 2|$$

holds for all  $\alpha \in \Sigma^+$ .

**Remark A.6.** Assume, as above, that  $m_\alpha > 0$  for  $\alpha \in \Sigma$ . Then the system of inequalities in Lemma A.5 admits the solution  $c = 0$  if and only if  $\Sigma$  is reduced and  $m_\alpha = 2$  for all  $\alpha \in \Sigma^+$ .

Let  $c \in [0, 1)$  obtained according to Lemma A.5 for some  $r > 1$ . The comparison of the recursion relations (A.11) and (A.14) is possible for a choice of  $H \in \mathfrak{a}$  so that  $\alpha(H) \geq \max\{2c\langle \rho(m), \alpha \rangle, 0\}$  for all  $\alpha \in \Sigma^+$ . This condition ensures that  $\mu(H) \geq 0$  for all  $\mu \in 2\Lambda$  and that the first summands on the right-hand side of (A.14) are nonnegative.

Arguments similar to those in Lemmas 4.7, 4.14 and 4.16 in [32], applied to the recursion relations (A.11) inductively on  $|I|$ , lead to the required estimates for the derivatives  $\partial_\lambda^I a_\mu(m; \mu)$  and hence to estimates for  $\partial_\lambda^I \Psi(m; \lambda, a)$  and  $\partial_\lambda^I \Phi(m; \lambda, a)$ .

**Lemma A.7.** *Suppose  $m \in \mathcal{M}^+$ . Let  $R > 0$  and let  $p_R$  be the polynomial defined in (A.8). Suppose  $c \in [0, 1)$  is chosen as in Lemma A.5 for some  $r > 1$ . Then, for every multi-index  $I$ , there is a constant  $K_{R,c,m,I} > 0$  such that*

$$|p_R(\lambda)^{|I|+1} \partial_\lambda^I a_\mu(m; \lambda)| \leq K_{R,c,m,I} d_\mu(m; c) (1 + |\lambda|)^{(|I|+1) \deg p_R}$$

for all  $\lambda \in \mathfrak{a}_{\mathbb{C}}^*(R)$  and all  $\mu \in 2\Lambda$ .

**Theorem A.8.** *Let  $m \in \mathcal{M}^+$ ,  $R > 0$  and let  $I$  be a multi-index. Let  $p_R$  denote the polynomial defined in (A.8). Then there exist  $c \in [0, 1)$  (depending only on the multiplicity function  $m$ ) and  $C_{R,c,m,I} > 0$  such that*

$$\begin{aligned}
|p_R(\lambda)^{(|I|+1)} \partial_\lambda^I \Psi(m; \lambda, a)| &\leq C_{R,c,m,I} \Delta(m; a)^{-c/2} (1 + |\log a|)^{|I|} \\
&\quad \times (1 + |\lambda|)^{(|I|+1) \deg p_R} e^{(c\rho(m) + \operatorname{Re} \lambda)(\log a)} \tag{A.15}
\end{aligned}$$

and

$$\begin{aligned} & |p_R(\lambda)^{(|I|+1)} \Delta(m; a)^{(c+1)/2} \partial_\lambda^I \Phi(m; \lambda, a)| \\ & \leq C_{R,c,m,I} (1 + |\log a|)^{|I|} \\ & \quad \times (1 + |\lambda|)^{(|I|+1) \deg p_R} e^{(c\rho(m) + \operatorname{Re} \lambda)(\log a)} \end{aligned} \quad (\text{A.16})$$

for all  $a \in A^+$  and  $\lambda \in \mathfrak{a}_{\mathbb{C}}^*(R)$ .

There is  $\omega > 0$  so that we can choose  $p_R \equiv 1$  for  $0 < R \leq \omega$ . In this case,

$$|\Delta(m; a)^{(c+1)/2} \partial_\lambda^I \Phi(m; \lambda, a)| \leq C_{R,c,m,I} (1 + |\log a|)^{|I|} e^{(c\rho(m) + \operatorname{Re} \lambda)(\log a)} \quad (\text{A.17})$$

for all  $a \in A^+$  and  $\lambda \in \mathfrak{a}_{\mathbb{C}}^*(R)$ .

**Proof.** Notice first that for  $J \leq I$

$$|\xi^J(\log a)| \leq |\log a|^{|J|} \leq (1 + |\log a|)^{|I|}.$$

Hence, because of (A.10), (A.13) and Lemma A.7,

$$\begin{aligned} & |p_R(\lambda)^{|I|+1} \partial_\lambda^I \Psi(m; \lambda, a)| \\ & = e^{\operatorname{Re} \lambda(\log a)} \sum_{\mu \in 2\Lambda} \sum_{J+K=I} \frac{I!}{J!K!} |\xi^J(\log a)| |p_R(\lambda)^{|I|+1} \partial_\lambda^K a_\mu(m; \lambda)| e^{-\mu(\log a)} \\ & \leq C_{R,c,m,I} \sum_{\mu \in 2\Lambda} d_\mu(m; c) e^{-\mu(\log a)} (1 + |\log a|)^{|I|} (1 + |\lambda|)^{(|I|+1) \deg p_R} e^{\operatorname{Re} \lambda(\log a)} \\ & \leq C_{R,c,m,I} \Delta_c(m; a) (1 + |\log a|)^{|I|} (1 + |\lambda|)^{(|I|+1) \deg p_R} e^{\operatorname{Re} \lambda(\log a)}. \end{aligned}$$

The first inequality therefore follows from (A.12). The last statement is a consequence of (A.1) and of the remark after (A.8).  $\square$

Lemma 4.13 is a special case of the following corollary to Theorem A.8.

**Corollary A.9.** Let  $m \in \mathcal{M}^+$  and let  $\omega$  be as in Theorem A.8. For every fixed  $\lambda \in \mathfrak{a}_{\mathbb{C}}^*(\omega)$  and every multi-index  $I$  the function  $\Delta(m; a) \partial_\lambda^I \Phi(m; \lambda, a)$  extends continuously on  $\overline{A^+}$  by setting it equal to zero on the boundary  $\partial(A^+)$  of  $A^+$ .

## Appendix B. Proof of Corollary 4.16

The ring  $\mathbb{C}[A_{\mathbb{C}}]$  is an integral domain and a UFD (see, e.g., [3], Chapter VI, §3, Lemma 1(i)). This in particular implies that irreducible and prime elements in  $\mathbb{C}[A_{\mathbb{C}}]$  coincide and that every finite set of elements of  $\mathbb{C}[A_{\mathbb{C}}]$  possesses a greatest common divisor.

**Lemma B.1.** *Let  $\Sigma$  be a reduced root system in  $\mathfrak{a}^*$ , and let  $\Delta = \Delta(1)$  denote the Weyl denominator.*

- (a) *For every  $\alpha \in \Sigma^+$  the elements  $1 + e^{-2\alpha}$  and  $1 - e^{-2\alpha}$  are relatively prime in  $\mathbb{C}[A_{\mathbb{C}}]$ .*  
 (b)  *$\prod_{\alpha \in \Sigma^+} \partial(A_{\alpha})(\Delta) \in \mathbb{C}[A_{\mathbb{C}}]$ , and  $\prod_{\alpha \in \Sigma^+} \partial(A_{\alpha})(\Delta)$  and  $1 - e^{-2\beta}$  are relatively prime for all  $\beta \in \Sigma^+$ .*

**Proof.** Suppose  $u \in \mathbb{C}[A_{\mathbb{C}}]$  divides  $1 + e^{-2\alpha}$  and  $1 - e^{-2\alpha}$ . Then there are  $f, g \in \mathbb{C}[A_{\mathbb{C}}]$  so that  $1 + e^{-2\alpha} = fu$  and  $1 - e^{-2\alpha} = gu$ . Thus  $2 = (1 + e^{-2\alpha}) + (1 - e^{-2\alpha}) = (f + g)u$ , which implies that  $u$  is a unit in  $\mathbb{C}[A_{\mathbb{C}}]$ . This proves (a).

For (b), let  $\beta_1, \dots, \beta_n$  be an enumeration of  $\Sigma^+$ , and set  $A_j = A_{\beta_j}$  for  $j = 1, \dots, n$ . Then

$$\partial(A_j)(\Delta) = e^{\rho(2)} \sum_{\alpha \in \Sigma^+} \langle \beta_j, \alpha \rangle (1 + e^{-2\alpha}) \prod_{\gamma \in \Sigma^+ \setminus \{\alpha\}} (1 - e^{-2\gamma}) \in \mathbb{C}[A_{\mathbb{C}}].$$

Hence  $\prod_{\alpha \in \Sigma^+} \partial(A_{\alpha})(\Delta) \in \mathbb{C}[A_{\mathbb{C}}]$ , and more precisely, we have:

$$\begin{aligned} \prod_{\alpha \in \Sigma^+} \partial(A_{\alpha})(\Delta) &= \prod_{j=1}^n \partial(A_j)(\Delta) \\ &= e^{n\rho(2)} \prod_{j=1}^n \sum_{k_j=1}^n \langle \beta_j, \beta_{k_j} \rangle (1 + e^{-2\beta_{k_j}}) \prod_{\gamma_{k_j} \in \Sigma^+ \setminus \{\beta_{k_j}\}} (1 - e^{-2\gamma_{k_j}}) \\ &= e^{n\rho(2)} \sum_{k_1=1}^n \dots \sum_{k_n=1}^n \prod_{j=1}^n \left[ \langle \beta_j, \beta_{k_j} \rangle (1 + e^{-2\beta_{k_j}}) \prod_{\gamma_{k_j} \in \Sigma^+ \setminus \{\beta_{k_j}\}} (1 - e^{-2\gamma_{k_j}}) \right]. \end{aligned}$$

Let  $\beta \in \Sigma^+$  be fixed. The only summand in which  $1 - e^{-2\beta}$  does not appear as a factor corresponds to  $\beta_{k_1} = \dots = \beta_{k_n} = \beta$ . Collecting the remaining terms together, we get for some  $f_{\beta} \in \mathbb{C}[A_{\mathbb{C}}]$ ,

$$\prod_{\alpha \in \Sigma^+} \partial(A_{\alpha})(\Delta) = (1 - e^{-2\beta}) f_{\beta} + e^{n\rho(2)} \left( \prod_{j=1}^n \langle \beta_j, \beta \rangle \right) (1 + e^{-2\beta})^n \prod_{\gamma \in \Sigma^+ \setminus \{\beta\}} (1 - e^{-2\gamma})^n.$$

By Lemma 4.11 and Part (a), the elements  $1 - e^{-2\beta}$  and  $(1 + e^{-2\beta})^n \prod_{\gamma \in \Sigma^+ \setminus \{\beta\}} (1 - e^{-2\gamma})^n$  are relatively prime. Thus also  $1 - e^{-2\beta}$  and  $\prod_{\alpha \in \Sigma^+} \partial(A_{\alpha})(\Delta)$  must be relatively prime.  $\square$

**Lemma B.2.** *Let  $\omega \in \mathbb{C}[A_{\mathbb{C}}]$ , and suppose  $\Delta$  does not divide  $\omega$  in  $\mathbb{C}[A_{\mathbb{C}}]$ . Then  $\Delta$  does not divide  $\omega \prod_{\alpha \in \Sigma^+} \partial(A_{\alpha})(\Delta)$  in  $\mathbb{C}[A_{\mathbb{C}}]$ .*

**Proof.** We prove that, if  $\Delta$  divides  $\omega \prod_{\alpha \in \Sigma^+} \partial(A_{\alpha})(\Delta)$  in  $\mathbb{C}[A_{\mathbb{C}}]$ , then  $\Delta$  must divide  $\omega$ . Let  $u \in \mathbb{C}[A_{\mathbb{C}}]$  be a prime dividing  $\Delta = e^{\rho(2)} \prod_{\beta \in \Sigma^+} (1 - e^{-2\beta})$ . Then  $u$  divides  $1 - e^{-2\beta}$

for some  $\beta \in \Sigma^+$ . Since  $1 - e^{-2\beta}$  and  $\prod_{\alpha \in \Sigma^+} \partial(A_\alpha)(\Delta)$  are relatively prime, so are also  $u$  and  $\prod_{\alpha \in \Sigma^+} \partial(A_\alpha)(\Delta)$ . But we are assuming that  $\Delta$ , hence  $u$ , divides  $\omega \prod_{\alpha \in \Sigma^+} \partial(A_\alpha)(\Delta)$ . So  $u$  divides  $\omega$ . Since  $\mathbb{C}[A_{\mathbb{C}}]$  is a UFD and  $u$  is an arbitrary prime dividing  $\Delta$ , we conclude that  $\Delta$  divides  $\omega$ .  $\square$

**Lemma B.3.** *Let  $\Delta^k \sum_I \omega_I \otimes \partial_a^I$  be the representation of an element of  $\mathbb{C}[A_{\mathbb{C}}] \otimes S(\mathfrak{a}_{\mathbb{C}})$  with  $\omega_I \in \mathbb{C}[A_{\mathbb{C}}]$  and  $k \in \mathbb{Z}$  maximal. If  $k \in -\mathbb{N}$ , then the differential operator*

$$\left( \prod_{\alpha \in \Sigma^+} \partial(A_\alpha)(\Delta) \right) \circ \left( \Delta^k \sum_I \omega_I \otimes \partial_a^I \right)$$

has singular coefficients on  $A_{\mathbb{C}}$ .

**Proof.** As in the proof of Lemma B.1(b), let us write  $\prod_{\alpha \in \Sigma^+} \partial(A_\alpha) = \prod_{j=1}^n \partial(A_j)$ . We first prove by induction on  $h \leq n$  that

$$\begin{aligned} \prod_{j=1}^h \partial(A_j) \circ \Delta^k \omega_I &= k(k-1) \cdots (k-h+1) \Delta^{k-h} \prod_{j=1}^h \partial(A_j)(\Delta) \omega_I \\ &\quad + \Delta^{k-h+1} \left[ \omega_{I,h} + \sum_{J(I,h)} \omega_{J(I,h)} \otimes \partial(A_{J(I,h)}) \right] \end{aligned} \quad (\text{B.1})$$

with  $\omega_{I,h}, \omega_{J(I,h)} \in \mathbb{C}[A_{\mathbb{C}}]$ , and where  $\partial(A_{J(I,h)}) := \partial(A_1)^{j_1} \cdots \partial(A_h)^{j_h}$  if  $J(I,h) = (j_1, \dots, j_h)$ . Indeed, for  $h=1$  we have:

$$\partial(A_1) \circ \Delta^k \omega_I = k \Delta^{k-1} \partial(A_1)(\Delta) \omega_I + \Delta^k [\partial(A_1)(\omega_I) + \omega_I \otimes \partial(A_1)].$$

Suppose inductively that (B.1) holds for  $h$ . Then

$$\begin{aligned} \prod_{j=1}^{h+1} \partial(A_j) \circ \Delta^k \omega_I &= \partial(A_{h+1}) \left[ k(k-1) \cdots (k-h+1) \Delta^{k-h} \prod_{j=1}^h \partial(A_j)(\Delta) \omega_I \right] \\ &\quad + \partial(A_{h+1}) \left[ \Delta^{k-h+1} \omega_{I,h} + \Delta^{k-h+1} \sum_{J(I,h)} \omega_{J(I,h)} \otimes \partial(A_{J(I,h)}) \right] \\ &= k(k-1) \cdots (k-h+1)(k-h) \Delta^{k-h-1} \prod_{j=1}^{h+1} \partial(A_j)(\Delta) \omega_I \\ &\quad + k(k-1) \cdots (k-h+1) \Delta^{k-h} \partial(A_{h+1}) \left[ \prod_{j=1}^h \partial(A_j)(\Delta) \omega_I \right] \\ &\quad + (k-h+1) \Delta^{k-h} \partial(A_{h+1})(\Delta) \omega_{I,h} + \Delta^{k-h+1} \partial(A_{h+1})(\omega_{I,h}) \end{aligned}$$

$$\begin{aligned}
& + (k-h+1)\Delta^{k-h}\partial(A_{h+1})(\Delta) \sum_{J(I,h)} \omega_{J(I,h)} \otimes \partial(A_{J(I,h)}) \\
& + \Delta^{k-h+1} \sum_{J(I,h)} \partial(A_{h+1})(\Delta) \omega_{J(I,h)} \otimes \partial(A_{J(I,h)}) \\
& + \Delta^{k-h+1} \sum_{J(I,h)} \omega_{J(I,h)} \otimes \partial(A_{h+1})\partial(A_{J(I,h)}) \\
& = k(k-1)\cdots(k-h)\Delta^{k-h-1} \prod_{j=1}^{h+1} \partial(A_j)(\Delta) \omega_I \\
& + \Delta^{k-h} \left[ \omega_{I,h+1} + \sum_{J(I,h+1)} \omega_{J(I,h+1)} \otimes \partial(A_{J(I,h+1)}) \right],
\end{aligned}$$

where

$$\begin{aligned}
\omega_{I,h+1} & := k(k-1)\cdots(k-h+1)\Delta^{k-h}\partial(A_{h+1}) \left[ \prod_{j=1}^h \partial(A_{h+1})(\Delta) \omega_I \right] \\
& + (k-h+1)\Delta^{k-h}\partial(A_{h+1})(\Delta) \omega_{I,h} + \Delta^{k-h+1}\partial(A_{h+1})(\omega_{I,h}) \in \mathbb{C}[A_{\mathbb{C}}]
\end{aligned}$$

and

$$\begin{aligned}
& \sum_{J(I,h+1)} \omega_{J(I,h+1)} \otimes \partial(A_{J(I,h+1)}) \\
& := (k-h+1)\partial(A_{h+1})(\Delta) \sum_{J(I,h)} \omega_{J(I,h)} \otimes \partial(A_{J(I,h)}) \\
& + \Delta \sum_{J(I,h)} \partial(A_{h+1})(\Delta) \omega_{J(I,h)} \otimes \partial(A_{J(I,h)}) \\
& + \Delta \sum_{J(I,h)} \omega_{J(I,h)} \otimes \partial(A_{h+1})\partial(A_{J(I,h)}) \in \mathbb{C}[A_{\mathbb{C}}] \otimes \mathbf{S}(\mathfrak{a}_{\mathbb{C}}).
\end{aligned}$$

Since  $k$  is maximal, there is a multi-index  $I_0$  such that  $\Delta$  does not divide  $\omega_{I_0}$ . Eq. (B.1) for  $h = n$  shows that the coefficient of  $\partial_a^{I_0}$  in  $\prod_{\alpha \in \Sigma^+} \partial(A_{\alpha})(\Delta) \circ \Delta^k \sum_I \omega_I \otimes \partial_a^I$  is equal to the sum of

$$k(k-1)\cdots(k-n+1)\Delta^{k-n} \prod_{\alpha \in \Sigma^+} \partial(A_{\alpha})(\Delta) \omega_{I_0} + \Delta^{k-n+1} \omega_{I_0,n}$$

and of those coefficients of  $\Delta^{k-n+1} \sum_{I \neq I_0} \sum_{J(I,n)} \omega_{J(I,n)} \otimes \partial(A_{J(I,n)}) \partial_a^I$  coming from the terms in  $\partial(A_{J(I,n)}) \partial_a^I$  which are equal to  $\partial_a^{I_0}$ . Consequently, the coefficient of  $\partial_a^{I_0}$  is of the form:

$$k(k-1)\cdots(k-n+1)\Delta^{k-n}\prod_{\alpha\in\Sigma^+}\partial(A_\alpha)(\Delta)\omega_{I_0}+\Delta^{k-n+1}\sigma_0$$

with  $\sigma_0 \in \mathbb{C}[A_{\mathbb{C}}]$ . By Lemma B.2,  $\Delta$  does not divide  $\prod_{\alpha\in\Sigma^+}\partial(A_\alpha)\omega_{I_0}$ . Thus the singularity of  $\Delta^{k-n}$  (with  $k-n < 0$ ) is neither cancelled by  $\prod_{\alpha\in\Sigma^+}\partial(A_\alpha)\omega_{I_0}$  nor by  $\Delta^{k-n+1}\sigma_0$  (in which a singularity of lower order appears).  $\square$

**Proof of Corollary 4.16.** Set  $G := \Delta(m)G_+(m-2; 2) \circ \Delta(-1)$ . Using the definition of  $D_m$  in (32) together with (29) and Example 4.8, we obtain:

$$\begin{aligned}\Delta(m)D_m &= \Delta(m)D_+(m) \\ &= \Delta(m)G_+(m-2; 2) \circ D_+(2) \\ &= \Delta(m)G_+(m-2; 2) \circ \tilde{\sigma}\Delta^{-1}\prod_{\alpha\in\Sigma^+}\partial(A_\alpha) \\ &= \tilde{\sigma}\left(G \circ \prod_{\alpha\in\Sigma^+}\partial(A_\alpha)\right).\end{aligned}$$

Hence

$$(\Delta(m)D_m)^* = \tilde{\sigma}(-1)^{|\Sigma^+|}\prod_{\alpha\in\Sigma^+}\partial(A_\alpha) \circ G^* = \sigma\prod_{\alpha\in\Sigma^+}\partial(A_\alpha) \circ G^*.$$

By Theorem 4.10,  $\Delta(m)D_m$ , and hence  $(\Delta(m)D_m)^*$ , belongs to  $\mathbb{C}[A_{\mathbb{C}}] \otimes \mathcal{S}(\mathfrak{a}_{\mathbb{C}})$ . Since  $G \in \mathbb{C}_{\Delta}[A_{\mathbb{C}}] \otimes \mathcal{S}(\mathfrak{a}_{\mathbb{C}})$ , so does  $G^*$ . Thus  $G^* = \Delta^k \sum_I \omega_I \otimes \partial_a^I$  with  $\omega_I \in \mathbb{C}[A_{\mathbb{C}}]$  and  $k \in \mathbb{Z}$  maximal. Since  $k \in -\mathbb{N}$  would imply  $(\Delta(m)D_m)^* \notin \mathbb{C}[A_{\mathbb{C}}] \otimes \mathcal{S}(\mathfrak{a}_{\mathbb{C}})$  by Lemma B.3, we conclude that  $k \in \mathbb{N}_0$ , i.e.,  $G^* \in \mathbb{C}[A_{\mathbb{C}}] \otimes \mathcal{S}(\mathfrak{a}_{\mathbb{C}})$ . Thus  $G \in \mathbb{C}[A_{\mathbb{C}}] \otimes \mathcal{S}(\mathfrak{a}_{\mathbb{C}})$ .  $\square$

### Appendix C. $K_\varepsilon$ -symmetric spaces with even multiplicities

In this appendix we report the infinitesimal classification of  $K_\varepsilon$ -symmetric spaces with even multiplicities by listing the  $K_\varepsilon$ -symmetric pairs  $(\mathfrak{g}, \mathfrak{h})$  with even multiplicities for which  $\mathfrak{g}$  is simple and noncompact. The list has been extracted from the classification due to Oshima and Sekiguchi [30]. It is presented in three tables respectively collecting (for the even multiplicity case) the Riemannian symmetric pairs (Table 1), the non-compactly causal (NCC) symmetric pairs (Table 2) and the other  $K_\varepsilon$  symmetric pairs (Table 3). A non-Riemannian  $K_\varepsilon$ -symmetric pair is said to be of type  $K_\varepsilon I$  if its signature  $\varepsilon$  comes from a gradation of first kind according to [19]. Otherwise it is said to be of type  $K_\varepsilon II$ . The symmetric pairs of type  $K_\varepsilon I$  coincide with the NCC symmetric pairs. Table 3 therefore collects all symmetric pairs with even multiplicities of type  $K_\varepsilon II$ .

The restricted root system  $\Sigma$  of a  $K_\varepsilon$ -symmetric pair with even multiplicities has at most two root lengths. The classification below shows that all multiplicities  $m_\alpha$  of  $\Sigma$  are equal, and moreover that they are all equal to 2 for symmetric pairs of type  $K_\varepsilon II$ . The

Table 1  
Riemannian symmetric pairs with even multiplicities

$\mathfrak{g}$	$\mathfrak{h} = \mathfrak{k}$	$\Sigma$	$m_\alpha$	
$\mathfrak{sl}(n, \mathbb{C})$	$\mathfrak{su}(n)$	$A_{n-1}$	2	$n \geq 2$
$\mathfrak{so}(2n+1, \mathbb{C})$	$\mathfrak{so}(2n+1)$	$B_n$	2	$n \geq 2$
$\mathfrak{sp}(n, \mathbb{C})$	$\mathfrak{sp}(n)$	$C_n$	2	$n \geq 3$
$\mathfrak{so}(2n, \mathbb{C})$	$\mathfrak{so}(2n)$	$D_n$	2	$n \geq 4$
$(\mathfrak{e}_6)_{\mathbb{C}}$	$\mathfrak{e}_6$	$E_6$	2	
$(\mathfrak{e}_7)_{\mathbb{C}}$	$\mathfrak{e}_7$	$E_7$	2	
$(\mathfrak{e}_8)_{\mathbb{C}}$	$\mathfrak{e}_8$	$E_8$	2	
$(\mathfrak{f}_4)_{\mathbb{C}}$	$\mathfrak{f}_4$	$F_4$	2	
$(\mathfrak{g}_2)_{\mathbb{C}}$	$\mathfrak{g}_2$	$G_2$	2	
$\mathfrak{su}^*(2n)$	$\mathfrak{sp}(n)$	$A_{n-1}$	4	$n \geq 2$
$\mathfrak{e}_{6(-26)}$	$\mathfrak{f}_{4(-20)}$	$A_2$	8	
$\mathfrak{so}(2n+1, 1)$	$\mathfrak{so}(2n+1)$	$A_1$	$2n$	$n \geq 3$

restricted root system and multiplicities of a  $K_\varepsilon$ -symmetric pair  $(\mathfrak{g}, \mathfrak{h})$  coincide with those of the corresponding Riemannian dual symmetric pair  $(\mathfrak{g}, \mathfrak{k})$ . They are explicitly reported in Tables 2 and 3 for the reader's convenience.

If  $\Sigma$  is of type  $X_n$  (with  $X_n \in \{A_n, B_n, C_n, \dots\}$ ), then the index  $n$  denotes the real rank of  $\mathfrak{g}$ . The range for  $n$  is chosen to avoid overlappings due to isomorphisms of symmetric spaces. These isomorphisms arise from isomorphisms of the lower-dimensional complex Lie algebras. We refer to [12], Chapter X, §6, for more information. The relevant symmetric pair isomorphisms are listed below.

Special isomorphisms of Riemannian symmetric spaces with even multiplicities (Table 1) are:

$$\begin{aligned}
 \mathfrak{so}(3, \mathbb{C}) &= \mathfrak{sp}(1, \mathbb{C}) \approx \mathfrak{sl}(2, \mathbb{C}), & \mathfrak{so}(3) &= \mathfrak{sp}(1) \approx \mathfrak{su}(2); \\
 \mathfrak{sp}(2, \mathbb{C}) &\approx \mathfrak{so}(5, \mathbb{C}), & \mathfrak{sp}(2) &\approx \mathfrak{so}(5); \\
 \mathfrak{so}(6, \mathbb{C}) &\approx \mathfrak{sl}(4, \mathbb{C}), & \mathfrak{so}(6) &\approx \mathfrak{su}(4); \\
 \mathfrak{so}(3, 1) &\approx \mathfrak{sl}(2, \mathbb{C}), & \mathfrak{so}(3) &\approx \mathfrak{su}(2); \\
 \mathfrak{so}(5, 1) &\approx \mathfrak{su}^*(4), & \mathfrak{so}(5) &\approx \mathfrak{sp}(2); \\
 \mathfrak{so}(8, \mathbb{C}) &= \mathfrak{so}(8, \mathbb{C}), & \mathfrak{so}^*(8) &\approx \mathfrak{so}(2, 6).
 \end{aligned}$$

The Lie algebra  $\mathfrak{so}(2, \mathbb{C})$  is not semisimple. Observe also that  $\mathfrak{so}(4, \mathbb{C}) \cong \mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C})$  is not simple. The structure of its homogeneous spaces can be therefore deduced from the structure of the homogeneous spaces of  $\mathfrak{sl}(2, \mathbb{C})$ .

In Table 2 we list all the non-compactly causal, or  $K_\varepsilon I$ , symmetric pairs with even multiplicities. The third column reports the subalgebra of  $\mathfrak{g}$  fixed by  $\theta\theta_\varepsilon$ , where  $\theta_\varepsilon$  is the involution associated with the  $K_\varepsilon$ -pair  $(\mathfrak{g}, \mathfrak{h})$ .

Special isomorphisms of NCC symmetric pairs with even multiplicities (Table 2) are:

$$\begin{aligned}
 \mathfrak{so}(3, \mathbb{C}) &= \mathfrak{sp}(1, \mathbb{C}) \approx \mathfrak{sl}(2, \mathbb{C}), & \mathfrak{so}(1, 2) &\approx \mathfrak{sp}(1, \mathbb{R}) \approx \mathfrak{su}(1, 1); \\
 \mathfrak{sp}(2, \mathbb{C}) &\approx \mathfrak{so}(5, \mathbb{C}), & \mathfrak{sp}(2, \mathbb{R}) &\approx \mathfrak{so}(3, 2); \\
 \mathfrak{so}(6, \mathbb{C}) &\approx \mathfrak{sl}(4, \mathbb{C}), & \mathfrak{so}(4, 2) &\approx \mathfrak{su}(2, 2); \\
 \mathfrak{so}(6, \mathbb{C}) &\approx \mathfrak{sl}(4, \mathbb{C}), & \mathfrak{so}^*(6) &\approx \mathfrak{su}(3, 1); \\
 \mathfrak{so}(3, 1) &\approx \mathfrak{sl}(2, \mathbb{C}), & \mathfrak{so}(2, 1) &\approx \mathfrak{su}(1, 1); \\
 \mathfrak{so}(5, 1) &\approx \mathfrak{su}^*(4), & \mathfrak{so}(4, 1) &\approx \mathfrak{sp}(1, 1).
 \end{aligned}$$



Table 2  
Non-compactly causal symmetric pairs with even multiplicities

$\mathfrak{g}$	$\mathfrak{h}$	$\mathfrak{g}^{\theta\theta_\varepsilon}$	$\Sigma$	$m_\alpha$	
$\mathfrak{sl}(n, \mathbb{C})$	$\mathfrak{su}(n-j, j)$	$\mathfrak{sl}(n-j, \mathbb{C}) \times \mathfrak{sl}(j, \mathbb{C}) \times \mathbb{C}$	$A_{n-1}$	2	$n \geq 2, 1 \leq j \leq [n/2]$
$\mathfrak{so}(2n+1, \mathbb{C})$	$\mathfrak{so}(2n-1, 2)$	$\mathfrak{so}(2n-1, \mathbb{C}) \times \mathbb{C}$	$B_n$	2	$n \geq 2$
$\mathfrak{sp}(n, \mathbb{C})$	$\mathfrak{sp}(n, \mathbb{R})$	$\mathfrak{gl}(n, \mathbb{C})$	$C_n$	2	$n \geq 3$
$\mathfrak{so}(2n, \mathbb{C})$	$\mathfrak{so}(2n-2, 2)$	$\mathfrak{so}(2n-2, \mathbb{C}) \times \mathbb{C}$	$D_n$	2	$n \geq 4$
$\mathfrak{so}(2n, \mathbb{C})$	$\mathfrak{so}^*(2n)$	$\mathfrak{gl}(n, \mathbb{C})$	$D_n$	2	$n \geq 5$
$(\mathfrak{e}_6)_\mathbb{C}$	$\mathfrak{e}_{6(-14)}$	$\mathfrak{so}(10, \mathbb{C}) \times \mathbb{C}$	$E_6$	2	
$(\mathfrak{e}_7)_\mathbb{C}$	$\mathfrak{e}_{7(-25)}$	$(\mathfrak{e}_6)_\mathbb{C} \times \mathbb{C}$	$E_7$	2	
$\mathfrak{su}^*(2n)$	$\mathfrak{sp}(n-j, j)$	$\mathfrak{su}^*(2(n-j)) \times \mathfrak{su}^*(2j) \times \mathbb{R}$	$A_{n-1}$	4	$n \geq 2, 1 \leq j \leq [n/2]$
$\mathfrak{e}_{6(-26)}$	$\mathfrak{f}_{4(-20)}$	$\mathfrak{so}(9, 1) \times \mathbb{R}$	$A_2$	8	
$\mathfrak{so}(2n+1, 1)$	$\mathfrak{so}(2n, 1)$	$\mathfrak{so}(2n+1) \times \mathbb{R}$	$A_1$	$2n$	$n \geq 3$

Table 3  
Other  $K_\varepsilon$ -symmetric pairs with even multiplicities

$\mathfrak{g}$	$\mathfrak{h}$	$\mathfrak{g}^{\theta\theta_\varepsilon}$	$\Sigma$	$m_\alpha$	
$\mathfrak{so}(2n+1, \mathbb{C})$	$\mathfrak{so}(2(n-j)+1, 2j)$	$\mathfrak{so}(2(n-j)+1, \mathbb{C}) \times \mathfrak{so}(2j, \mathbb{C})$	$B_n$	2	$n \geq 2, 2 \leq j \leq n$
$\mathfrak{sp}(n, \mathbb{C})$	$\mathfrak{sp}(n-j, j)$	$\mathfrak{sp}(n-j, \mathbb{C}) \times \mathfrak{sp}(j, \mathbb{C})$	$C_n$	2	$n \geq 3, 1 \leq j \leq [n/2]$
$\mathfrak{so}(2n, \mathbb{C})$	$\mathfrak{so}(2(n-j), 2j)$	$\mathfrak{so}(2(n-j), \mathbb{C}) \times \mathfrak{so}(2j, \mathbb{C})$	$D_n$	2	$n \geq 4, 2 \leq j \leq [n/2]$
$(\mathfrak{e}_6)_\mathbb{C}$	$\mathfrak{e}_{6(2)}$	$\mathfrak{sl}(6, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C})$	$E_6$	2	
$(\mathfrak{e}_7)_\mathbb{C}$	$\mathfrak{e}_{7(7)}$	$\mathfrak{sl}(8, \mathbb{C})$	$E_7$	2	
$(\mathfrak{e}_7)_\mathbb{C}$	$\mathfrak{e}_{7(-5)}$	$\mathfrak{so}(12, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C})$	$E_7$	2	
$(\mathfrak{e}_8)_\mathbb{C}$	$\mathfrak{e}_{8(8)}$	$\mathfrak{so}(16, \mathbb{C})$	$E_8$	2	
$(\mathfrak{e}_8)_\mathbb{C}$	$\mathfrak{e}_{8(-24)}$	$\mathfrak{e}_{7\mathbb{C}} \times \mathfrak{sl}(2, \mathbb{C})$	$E_8$	2	
$(\mathfrak{f}_4)_\mathbb{C}$	$\mathfrak{f}_{4(4)}$	$\mathfrak{sp}(3, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C})$	$F_4$	2	
$(\mathfrak{f}_4)_\mathbb{C}$	$\mathfrak{f}_{4(-20)}$	$\mathfrak{so}(9, \mathbb{C})$	$F_4$	2	
$(\mathfrak{g}_2)_\mathbb{C}$	$\mathfrak{g}_{2(2)}$	$\mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C})$	$G_2$	2	

Table 3 contains all the other  $K_\varepsilon$ -symmetric pairs, i.e., those of type  $K_\varepsilon II$ , with even multiplicities.

A special isomorphism of  $K_\varepsilon II$  symmetric pairs with even multiplicities is:

$$\mathfrak{sp}(2, \mathbb{C}) \approx \mathfrak{so}(5, \mathbb{C}), \quad \mathfrak{sp}(1, 1) \approx \mathfrak{so}(1, 4).$$

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